

Lecture Notes

Schrödinger Equations on Hyperbolic Spaces and Beyond

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I Introduction

The Schrödinger equation is one of the fundamental examples of a *dispersive equation*. Here dispersion refers to the phenomenon that a localized initial wave tends to spread out in space as time evolves. In the Euclidean space $\mathbb{R}^n$, the *free* Schrödinger equation is

$$i\partial_t u(t, x) + \Delta_x u(t, x) = 0, \quad u(0, x) = u_0(x), \quad (1.1)$$

where $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ is a complex-valued function and Δ is the Euclidean Laplacian. The solution describes the time evolution of the initial state u_0 and is given by

$$u(t, x) = e^{it\Delta} u_0(x) = (s_t * u_0)(x).$$

Here s_t is the convolution kernel of the free Schrödinger propagator $e^{it\Delta}$, equivalently the fundamental solution of the free Schrödinger equation (1.1). By Fourier analysis, we obtain its explicit formula

$$s_t(x) = (4\pi it)^{-\frac{n}{2}} e^{\frac{it|x|^2}{4t}}, \quad t \neq 0. \quad (1.2)$$

In quantum mechanics, u is interpreted as a wave function, $|u(t, x)|^2$ represents the probability density of finding a particle near x at time t , and $s_t(x - y)$ describes how the initial wave at y contributes to the wave function at x after time t .

While the linear equation captures the basic dispersive mechanism, many important questions arise for nonlinear Schrödinger equations, where the linear dispersive evolution interacts with a nonlinear self-interaction term. In these lectures, our main nonlinear questions will be *well-posedness* and *scattering*. Roughly speaking, well-posedness asks whether the equation admits a unique solution depending continuously on the initial data, while scattering asks whether a global solution approaches a free Schrödinger evolution as $t \rightarrow \pm\infty$.

Dispersive and Strichartz estimates serve as a bridge between the linear theory and nonlinear problems: they convert the spreading properties of the linear Schrödinger flow into spacetime bounds that can be used to control nonlinear interactions. *Dispersive estimates* are fixed-time decay estimates for the linear propagator. In the Euclidean case, the explicit kernel formula (1.2) directly yields

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n)} \leq C|t|^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}. \quad (1.3)$$

By interpolating this estimate with the conservation of the L^2 norm, one obtains the family of estimates

$$\|e^{it\Delta} f\|_{L^q(\mathbb{R}^n)} \leq C|t|^{-n(\frac{1}{2} - \frac{1}{q})} \|f\|_{L^{q'}(\mathbb{R}^n)}, \quad 2 \leq q \leq \infty, \quad (1.4)$$

where q' is the dual exponent of q . These estimates express the fact that the linear flow sends concentrated data into increasingly spread-out waves.

Strichartz estimates are spacetime estimates for the same linear flow. Instead of measuring the solution at a fixed time, they control mixed norms in both time and space, typically of the form ¹

$$\|e^{it\Delta} u_0\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |e^{it\Delta} u_0(x)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} \leq C \|u_0\|_{L^2(\mathbb{R}^n)} \quad (1.5)$$

¹When $p = \infty$ or $q = \infty$, the mixed norm is understood with the usual modification.

for p and q satisfying the *admissible* condition:

$$2 \leq p, q \leq \infty, \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (p, q, n) \neq (2, \infty, 2). \quad (1.6)$$

The last condition excludes the two-dimensional endpoint.



Figure 1: Admissible sets in dimension $n \geq 3$ and $n = 2$.

From the viewpoint of regularity, Strichartz estimates are stronger than what one obtains by applying Sobolev embedding at each fixed time. Sobolev embedding gives spatial integrability only after paying spatial derivatives, whereas Strichartz estimates use the dispersive evolution itself to produce spacetime integrability from the conserved L^2 norm. This absence of extra derivative loss is especially important near scaling-critical regularities and in scattering problems.

Exercise 1: Homogeneous Strichartz estimate

The goal of this exercise is to prove the homogeneous Strichartz estimate (1.5) in the non-endpoint case where $2 < p < \infty$.

1. Use the Fourier transform and its inversion formula to prove the explicit formula (1.2) for the convolution kernel s_t . Verify that s_t is the fundamental solution of the equation (1.1).
2. Derive the $L^1 \rightarrow L^\infty$ estimate (1.3), and prove the conservation of the L^2 norm for the Schrödinger propagator $e^{it\Delta}$. Then use interpolation to prove the $L^{q'} \rightarrow L^q$ estimate (1.4) for $2 \leq q \leq \infty$.
3. Let $T = e^{it\Delta}$, and let T^* be its dual operator:

$$Tf(t, x) = e^{it\Delta}f(x) \quad \text{and} \quad T^*F(x) = \int_{\mathbb{R}} e^{-is\Delta}F(s, x) \, ds.$$

Prove the equivalence between the boundedness of

$$\begin{aligned} T &: L_x^2(\mathbb{R}^n) \longrightarrow L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n), \\ T^* &: L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{R}^n) \longrightarrow L_x^2(\mathbb{R}^n), \\ TT^* &: L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{R}^n) \longrightarrow L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n). \end{aligned}$$

4. Prove that, for every pair (p, q) satisfying the admissible condition (1.6) with $2 < p < \infty$, the operator TT^* is bounded from $L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{R}^n)$ to $L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n)$. Then, deduce the Strichartz estimate (1.5). (Hint. Use the one-dimensional Hardy–Littlewood–Sobolev inequality.)

It is intuitively clear that changing the geometric structure of the underlying space should affect the way waves spread. Hyperbolic spaces provide a fundamental setting for understanding these phenomena: they are the simplest non-Euclidean spaces of constant negative curvature, while still possessing a rich and explicit harmonic analysis. Their exponential volume growth, the spectral gap of the Laplace–Beltrami operator, and the structure of spherical Fourier analysis change the behavior of the Schrödinger kernel and lead to dispersive and Strichartz estimates that differ substantially from their Euclidean counterparts. One of the main objectives of this lecture is therefore to explain how harmonic analysis on hyperbolic spaces can play the role of the Euclidean Fourier analysis framework and be used to derive pointwise estimates for the Schrödinger kernel.

II Harmonic Analysis on Hyperbolic Spaces

The purpose of this section is to set up the analytic framework used in the study of Schrödinger equations on hyperbolic spaces. We begin with the elementary Riemannian geometry of $\mathbb{H}^n$: its metric, distance function, volume element, volume growth, and Laplace–Beltrami operator. We then regard $\mathbb{H}^n$ as a non-compact symmetric space of rank one. This point of view provides the group-theoretic structure underlying spherical Fourier analysis. Finally, we introduce the spherical Fourier transform, together with the basic properties of spherical functions and the Harish–Chandra $\mathbf{c}$ -function.

2.1 Geometry of hyperbolic spaces

For $n \geq 2$, the real hyperbolic space $\mathbb{H}^n$ is characterized, up to isometry, as the unique simply connected complete n -dimensional Riemannian manifold with constant sectional curvature -1 . There are several equivalent models of hyperbolic space, which are different realizations, or different coordinate descriptions, of the same Riemannian manifold $\mathbb{H}^n$. The most common models include the hyperboloid model, the Poincaré ball model, and the upper half-space model. Each model has its own advantages, and different models are useful for understanding different questions.

2.1.1 Hyperboloid model

The *hyperboloid* model is a convenient model for our purposes in these lectures. In this model, $\mathbb{H}^n$ is realized as the upper sheet of the two-sheeted hyperboloid

$$\mathbb{H}^n = \{x \in \mathbb{R}^{1+n} : L(x, x) = -1, x_0 > 0\},$$

where the *Lorentz bilinear form* is given by

$$L(x, y) = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

The tangent space at $x \in \mathbb{H}^n$ is

$$T_x \mathbb{H}^n = \{v \in \mathbb{R}^{1+n} : L(x, v) = 0\}.$$

The restriction of L to $T_x \mathbb{H}^n$ is positive definite, and hence defines the Riemannian metric on $\mathbb{H}^n$.

If $\gamma : [a, b] \rightarrow \mathbb{H}^n$ is a piecewise C^1 curve, its length is

$$\ell(\gamma) = \int_a^b \sqrt{L(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

For $x, y \in \mathbb{H}^n$, the *Riemannian distance* is defined by

$$d(x, y) = \inf \{ \ell(\gamma) : \gamma(a) = x, \gamma(b) = y \},$$

where the infimum is taken over all piecewise C^1 curves in $\mathbb{H}^n$ joining x and y . In the hyperboloid model, this distance admits the following explicit formula.

Proposition 2.1. *For any $x, y \in \mathbb{H}^n$,*

$$d(x, y) = \operatorname{arcosh}(-L(x, y)),$$

where arcosh denotes the inverse of $\cosh : [0, \infty) \rightarrow [1, \infty)$.

Exercise 2: Hyperbolic functions

Recall the definitions of the hyperbolic sine and cosine functions:

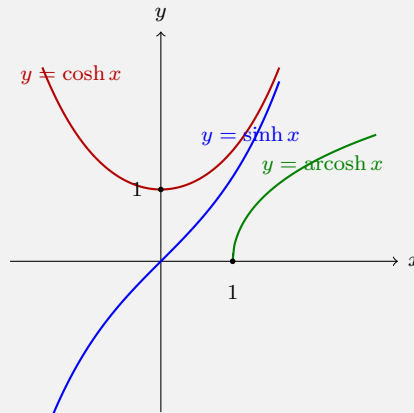
$$\sinh r = \frac{e^r - e^{-r}}{2}, \quad \cosh r = \frac{e^r + e^{-r}}{2}, \quad r \in \mathbb{R}.$$

- (i) Check that $\sinh r$ is odd and strictly increasing on $\mathbb{R}$; $\cosh r$ is even, strictly decreasing on $(-\infty, 0]$, and strictly increasing on $[0, \infty)$.
- (ii) Check the elementary identities

$$\cosh^2 r - \sinh^2 r = 1, \quad \frac{d}{dr} \sinh r = \cosh r, \quad \frac{d}{dr} \cosh r = \sinh r.$$

- (iii) For $x \geq 1$, check that $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$ and

$$\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2 - 1}} \quad x > 1.$$



Proof of Proposition 2.1. We first note that $-L(x, y) \geq 1$ for all $x, y \in \mathbb{H}^n$, and the equality holds only when $x = y$. Indeed, write $x = (x_0, x')$ and $y = (y_0, y')$, with $x_0, y_0 > 0$ and $x', y' \in \mathbb{R}^n$. Since $x, y \in \mathbb{H}^n$,

$$x_0^2 - |x'|^2 = 1, \quad y_0^2 - |y'|^2 = 1.$$

Thus

$$-L(x, y) = x_0 y_0 - x' \cdot y' \geq x_0 y_0 - |x'| |y'| = \sqrt{1 + |x'|^2} \sqrt{1 + |y'|^2} - |x'| |y'| \geq 1,$$

because

$$(1 + |x'|^2)(1 + |y'|^2) - (1 + |x'| |y'|)^2 = (|x'| - |y'|)^2 \geq 0.$$

Therefore $\operatorname{arcosh}(-L(x, y))$ is well defined for any $x, y \in \mathbb{H}^n$.

For simplicity, we set $A = -L(x, y)$ and $r = \operatorname{arcosh} A$. We will prove $d(x, y) = r$. If $x = y$, then $A = 1$ and $r = 0$, so the claim is immediate. Assume henceforth that $x \neq y$, so that $A > 1$. Define

$$u = \frac{y - Ax}{\sqrt{A^2 - 1}}.$$

This vector is not chosen arbitrarily: it is the unit tangent direction at x determined by y . In Euclidean space, the unit direction from x to y is $(y - x)/|y - x|$. On the hyperboloid, however, $y - x$ generally does not belong to the tangent space. We therefore decompose y into its normal and tangential components at x :

$$y = Ax + (y - Ax).$$

Since $A = -L(x, y)$ and $L(x, x) = -1$,

$$L(x, y - Ax) = L(x, y) - AL(x, x) = -A + A = 0.$$

Thus $y - Ax \in T_x \mathbb{H}^n$ is the tangential component of y at x . Moreover,

$$L(y - Ax, y - Ax) = A^2 - 1.$$

Dividing this vector by $\sqrt{A^2 - 1}$ therefore gives the unit tangent vector u .

Using equalities $L(x, x) = L(y, y) = -1$ and $L(x, y) = -A$, a direct computation gives $L(x, u) = 0$ and $L(u, u) = 1$. In other words, u belongs to $T_x \mathbb{H}^n$ and has unit length. Now define the curve

$$\gamma_u(t) = \cosh t x + \sinh t u, \quad 0 \leq t \leq r,$$

with endpoints $\gamma_u(0) = x$ and $\gamma_u(r) = Ax + \sqrt{A^2 - 1} u = y$. From the facts

$$L(\gamma_u(t), \gamma_u(t)) = -\cosh^2 t + \sinh^2 t = -1,$$

and

$$L(\dot{\gamma}_u(t), \dot{\gamma}_u(t)) = -\sinh^2 t + \cosh^2 t = 1,$$

we deduce that γ_u defines a curve on $\mathbb{H}^n$ and has unit speed, hence length r . As a consequence, we obtain $d(x, y) \leq r$.

Now, we turn to the reverse inequality. Let $\gamma : [0, 1] \rightarrow \mathbb{H}^n$ be any piecewise C^1 curve joining x to y , and set

$$F(t) = -L(x, \gamma(t)).$$

By the first part, $F(t) \geq 1$ for all t , while $F(0) = 1$ and $F(1) = A$. On each interval where γ is C^1 , differentiating $L(\gamma(t), \gamma(t)) = -1$ gives $L(\gamma(t), \dot{\gamma}(t)) = 0$. Define

$$w(t) = x - F(t)\gamma(t).$$

Then $w(t) \in T_{\gamma(t)}\mathbb{H}^n$. In fact, $w(t)$ is the tangential component of the fixed vector x at the point $\gamma(t)$. A direct computation gives $L(w(t), w(t)) = F(t)^2 - 1$. Furthermore,

$$F'(t) = -L(x, \dot{\gamma}(t)) = -L(w(t), \dot{\gamma}(t)).$$

Since the restriction of L to each tangent space is positive definite, the Cauchy–Schwarz inequality on $T_{\gamma(t)}\mathbb{H}^n$ gives

$$|F'(t)| \leq \sqrt{F(t)^2 - 1} \sqrt{L(\dot{\gamma}(t), \dot{\gamma}(t))}.$$

Therefore²,

$$\left| \frac{d}{dt} \operatorname{arcosh}(F(t)) \right| \leq \frac{|F'(t)|}{\sqrt{F(t)^2 - 1}} \leq \sqrt{L(\dot{\gamma}(t), \dot{\gamma}(t))}.$$

Integrating from 0 to 1 yields

$$\operatorname{arcosh}(A) - \operatorname{arcosh}(1) \leq \ell(\gamma).$$

In other words, we obtained $r = \operatorname{arcosh} A \leq \ell(\gamma)$. Taking the infimum over all such curves γ gives $r \leq d(x, y)$. The proof is therefore complete. $\square$

2.1.2 Geodesic polar coordinates

The proof of Proposition 2.1 suggests a general description of geodesic polar coordinates in the hyperboloid model. Fix a center $y \in \mathbb{H}^n$. For every $x \in \mathbb{H}^n$ with $x \neq y$, there exist $r > 0$ and a unit tangent vector $\omega \in T_y\mathbb{H}^n$ such that

$$x = \cosh r y + \sinh r \omega, \quad L(\omega, \omega) = 1, \quad r = d(x, y).$$

Taking $y = o$ gives the following geodesic polar coordinates of $x \in \mathbb{H}^n$.

Corollary 2.2. *For every $x \in \mathbb{H}^n$, there exists $\omega \in \mathbb{S}^{n-1}$ such that $x = (\cosh r, \sinh r \omega)$, where $r = d(o, x)$ denotes the Riemannian distance between x and the origin $o = (1, 0, \dots, 0)$ of $\mathbb{H}^n$.*

Having identified the radial and angular variables in the hyperboloid model, we now express the Riemannian metric in terms of these geodesic polar coordinates.

²In these lecture notes we shall occasionally use formal computations in order to keep the exposition concise. Such computations can be justified by standard approximation or regularization arguments. Here, for example, one may replace $F(t)$ by $F(t) + \varepsilon$, perform the calculation for $\varepsilon > 0$, and then let $\varepsilon \downarrow 0$. Later computations with fundamental solutions and oscillatory kernels will similarly be understood in the sense of distributions when needed.

Corollary 2.3. *The Riemannian metric on $\mathbb{H}^n$ is given by*

$$g_{\mathbb{H}^n} = (dr)^2 + \sinh^2 r d\omega^2. \quad (2.1)$$

Proof. In geodesic polar coordinates, we write $x_0 = \cosh r$ and $x' = \sinh r \omega$, where $\omega \in \mathbb{S}^{n-1}$. Hence

$$dx_0 = \sinh r dr,$$

and

$$dx' = d(\sinh r \omega) = \cosh r dr \omega + \sinh r d\omega.$$

Since $\omega \in \mathbb{S}^{n-1}$, we have $|\omega|^2 = 1$ and $\omega \cdot d\omega = 0$. Therefore,

$$\begin{aligned} |dx'|^2 &= \cosh^2 r (dr)^2 |\omega|^2 + 2 \sinh r \cosh r dr (\omega \cdot d\omega) + \sinh^2 r |d\omega|^2 \\ &= \cosh^2 r (dr)^2 + \sinh^2 r d\omega^2 \end{aligned}$$

Substituting the above expressions gives

$$L(dx, dx) = -(dx_0)^2 + |dx'|^2 = -\sinh^2 r (dr)^2 + \cosh^2 r (dr)^2 + \sinh^2 r d\omega^2.$$

Using $\cosh^2 r - \sinh^2 r = 1$, we obtain

$$L(dx, dx) = (dr)^2 + \sinh^2 r d\omega^2.$$

□

Once the metric is known in geodesic polar coordinates, the corresponding Riemannian volume element can be read from the determinant of its metric matrix. Since each of the $n - 1$ angular directions is scaled by $\sinh r$, the resulting Jacobian contains the factor $(\sinh r)^{n-1}$.

Corollary 2.4. *In geodesic polar coordinates, the Riemannian measure on $\mathbb{H}^n$ is given by*

$$d\mu_{\mathbb{H}^n} = (\sinh r)^{n-1} dr d\omega,$$

where $d\omega$ denotes the standard Riemannian measure on $\mathbb{S}^{n-1}$.

Proof. Let $\theta = (\theta^1, \dots, \theta^{n-1})$ be local coordinates on $\mathbb{S}^{n-1}$, and write the standard metric on $\mathbb{S}^{n-1}$ as

$$d\omega^2 = h_{ab}(\theta) d\theta^a d\theta^b,$$

where

$$h_{ab}(\theta) = \frac{\partial \omega}{\partial \theta^a} \cdot \frac{\partial \omega}{\partial \theta^b}, \quad 1 \leq a, b \leq n-1.$$

These functions are the matrix coefficients in the local coordinates θ . We deduce from (2.1) that the metric matrix of $g_{\mathbb{H}^n}$ has block diagonal form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 r h_{ab}(\theta) \end{pmatrix}.$$

Recall that if a Riemannian metric is written in local coordinates as $g = g_{ij} dx^i dx^j$, then its volume element is

$$d\mu_g = \sqrt{\det(g_{ij})} dx^1 \cdots dx^n.$$

In our coordinates, the determinant of the block diagonal matrix is

$$\det(g_{ij}) = (\sinh^2 r)^{n-1} \det(h_{ab}).$$

It follows that

$$d\mu_{\mathbb{H}^n} = (\sinh r)^{n-1} \sqrt{\det(h_{ab})} dr d\theta^1 \cdots d\theta^{n-1} = (\sinh r)^{n-1} dr d\omega.$$

The formula is valid away from the base point o ; the point $r = 0$ is the usual polar-coordinate singularity and has measure zero. $\square$

Remark 2.5. The radial factor in the volume element has two different asymptotic regimes:

$$(\sinh r)^{n-1} \asymp \begin{cases} r^{n-1} & r \rightarrow 0, \\ e^{(n-1)r} & r \rightarrow +\infty. \end{cases}$$

Therefore, the volume $d\mu_{\mathbb{H}^n}$ shows the Euclidean nature around the origin, while the exponential growth at infinity is one of the basic geometric features of hyperbolic space. Consequently, the volume of a geodesic ball satisfies

$$\text{Vol}(B_{\mathbb{H}^n}(o, R)) = |\mathbb{S}^{n-1}| \int_0^R (\sinh r)^{n-1} dr \sim \frac{|\mathbb{S}^{n-1}|}{2^{n-1}(n-1)} e^{(n-1)R} \quad \text{as } R \rightarrow \infty.$$

2.1.3 Laplace–Beltrami operator

We next introduce the Laplace–Beltrami operator on $\mathbb{H}^n$. Let (M, g) be a Riemannian manifold. The Laplace–Beltrami operator is the intrinsic second order differential operator $\Delta_g = \text{div}_g \nabla_g$. In the local coordinates $x = (x^1, \dots, x^n)$, we write

$$g = g_{ij} dx^i dx^j, \quad g_{ij} = g(\partial_i, \partial_j),$$

and denote by (g^{ij}) the inverse matrix of (g_{ij}) . Then the Laplace–Beltrami operator is written as

$$\Delta_g f = \frac{1}{\sqrt{\det(g_{ij})}} \partial_i \left(\sqrt{\det(g_{ij})} g^{ij} \partial_j f \right), \quad f \in C^\infty(M).$$

Proposition 2.6. *In geodesic polar coordinates,*

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1) \coth r \partial_r + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{n-1}}.$$

If f is radial, that is, if $f(x) = F(r)$ depends only on $r = d(o, x)$, then

$$\Delta_{\mathbb{H}^n} f = F''(r) + (n-1) \coth r F'(r).$$

Proof. Using the notations from the proof of Corollary 2.4, we have

$$g_{\mathbb{H}^n} = (dr)^2 + \sinh^2 r d\omega^2,$$

with $d\omega^2 = h_{ab}(\theta) d\theta^a d\theta^b$. Here, the functions h_{ab} are the coefficients of the round metric in the local coordinates $\theta = (\theta^1, \dots, \theta^{n-1})$ on $\mathbb{S}^{n-1}$. Since the metric contains no mixed terms $dr d\theta^a$, the radial direction is orthogonal to the angular directions, and

$$g_{rr} = 1, \quad g_{ra} = 0, \quad g_{ab} = \sinh^2 r h_{ab}(\theta).$$

Hence

$$g^{rr} = 1, \quad g^{ra} = 0, \quad g^{ab} = \frac{1}{\sinh^2 r} h^{ab},$$

where (h^{ab}) is the inverse matrix of (h_{ab}) . Recall that

$$\sqrt{\det(g_{ij})} = (\sinh r)^{n-1} \sqrt{\det(h_{ab})}.$$

Applying the local formula for the Laplace–Beltrami operator gives

$$\begin{aligned} \Delta_{\mathbb{H}^n} f &= \frac{1}{(\sinh r)^{n-1} \sqrt{\det(h_{ab})}} \partial_r \left((\sinh r)^{n-1} \sqrt{\det(h_{ab})} \partial_r f \right) \\ &\quad + \frac{1}{(\sinh r)^{n-1} \sqrt{\det(h_{ab})}} \partial_a \left((\sinh r)^{n-1} \sqrt{\det(h_{ab})} \frac{1}{\sinh^2 r} h^{ab} \partial_b f \right). \end{aligned}$$

On the one hand, the factor $\sqrt{\det(h_{ab})}$ is independent of r , so the radial part is

$$\frac{1}{(\sinh r)^{n-1}} \partial_r \left((\sinh r)^{n-1} \partial_r f \right) = \partial_r^2 f + (n-1) \coth r \partial_r f.$$

On the other hand, since $\sinh r$ is independent of the angular variables, the angular part is

$$\frac{1}{\sinh^2 r} \frac{1}{\sqrt{\det(h_{ab})}} \partial_a \left(\sqrt{\det(h_{ab})} h^{ab} \partial_b f \right) = \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{n-1}} f.$$

Combining the radial and angular contributions proves the formula. For a radial function, the angular term vanishes, and the stated radial formula follows. $\square$

Proposition 2.7. *Let $\rho = (n-1)/2$. Then the nonnegative self-adjoint operator $-\Delta_{\mathbb{H}^n}$ on $L^2(\mathbb{H}^n, d\mu_{\mathbb{H}^n})$ has spectrum*

$$\text{spec}(-\Delta_{\mathbb{H}^n}) = [\rho^2, \infty).$$

Remark 2.8. This spectral statement will follow naturally from spherical Fourier analysis: under the spherical Fourier transform, $-\Delta_{\mathbb{H}^n}$ becomes multiplication by $\lambda^2 + \rho^2$.

2.2 Hyperbolic space as a symmetric space

So far we have studied hyperbolic space as a Riemannian manifold. This gives the metric, the distance function, the volume element, and the Laplace–Beltrami operator. For harmonic analysis, however, one also needs to understand why $\mathbb{H}^n$ has enough symmetry to admit a useful analogue of the Euclidean Fourier transform. From this point of view, the main idea is that points of $\mathbb{H}^n$ can be moved around by isometries in a very controlled way. Once a base point is fixed, the isometries preserving that point describe the angular symmetry, while the remaining direction records the distance from the base point. In the language introduced below, this will be expressed by the realization of $\mathbb{H}^n$ as a homogeneous space G/K and by the Cartan decomposition $G = KAK$. These structures provide the natural bridge from the geometry developed above to spherical Fourier analysis.

2.2.1 Isometry group and homogeneous space

Before continuing our discussion of hyperbolic space, we first introduce what is meant by a homogeneous space. Intuitively, a manifold is homogeneous if its symmetries can move any point to any other point. In the Riemannian setting, the symmetries are isometries. We recall the basic terminology and then illustrate it with two familiar examples.

Definition 2.9. *Let (M, g) be a Riemannian manifold. The isometry group of (M, g) is the group, under composition, of all diffeomorphisms on M that preserve the Riemannian metric:*

$$\text{Isom}(M) = \left\{ \Phi : M \rightarrow M \mid \underbrace{\Phi \text{ is smooth and bijective, } \Phi^{-1} \text{ is smooth, } \Phi^*g = g}_{\Phi \text{ is a diffeomorphism}} \right\}.$$

- The identity component $\text{Isom}_0(M)$ is the connected component of $\text{Isom}(M)$ containing the identity map id_M .
- A Riemannian manifold M is called a homogeneous space if its isometry group acts transitively. That is, for any two points $x, y \in M$, there exists $\Phi \in \text{Isom}(M)$ such that $\Phi(x) = y$.
- A Riemannian manifold M is called a symmetric space if M is connected, homogeneous, and there exists an involutive isometry of M such that it has at least one isolated fixed point.

Example 2.10. Consider $\mathbb{R}^3$ with the standard Euclidean metric. In this case, the isometry group is

$$\text{Isom}(\mathbb{R}^3) = \{ \Phi_{A,b} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \Phi_{A,b}(x) = Ax + b, A \in O(3), b \in \mathbb{R}^3 \} \simeq O(3) \ltimes \mathbb{R}^3.$$

Here A is a 3×3 orthogonal matrix, so it preserves Euclidean lengths and angles, while b is a vector in $\mathbb{R}^3$ and represents a translation. Thus an element of $O(3) \ltimes \mathbb{R}^3$ acts on $\mathbb{R}^3$ by first applying the orthogonal transformation A and then translating by b .

The full isometry group is not connected. Recall that a topological space M is called connected if it cannot be written as the union of two nonempty disjoint open subsets. For Lie groups, connected components can equivalently be understood in terms of continuous paths. The determinant of the linear part $A \in O(3)$ is a continuous function, and on $O(3)$ it can take only the values ± 1 . Hence an

isometry with $\det A = 1$ cannot be joined by a continuous path to one with $\det A = -1$. Therefore, the identity component consists precisely of the orientation-preserving isometries:

$$G := \text{Isom}_0(\mathbb{R}^3) \simeq \text{SO}(3) \ltimes \mathbb{R}^3.$$

For our purposes, it is enough to work with the identity component, since it already acts transitively on $\mathbb{R}^3$ and has the same Lie algebra as the full isometry group. The other connected components are obtained from the identity component by multiplication by discrete isometries. $\mathbb{R}^3$ is clearly homogeneous under the action of G , since any point can be moved to any other by a rotation followed by a translation.

Intuitively, we describe a point of $\mathbb{R}^3$ by choosing an isometry which sends the origin to that point. This choice is not unique: many isometries may send the origin to the same point. Two such isometries differ precisely by an isometry fixing the base point, namely by a rotation. Therefore, in order to recover the space itself from the group G , we should identify isometries which differ only by an element fixing the base point. These elements form the stabilizer subgroup

$$K := G_o = \{\Phi \in G : \Phi(0) = 0\} \simeq \text{SO}(3).$$

Thus, from the symmetric space point of view, $\mathbb{R}^3$ can be written as

$$\mathbb{R}^3 \simeq (\text{SO}(3) \ltimes \mathbb{R}^3) / \text{SO}(3).$$

The canonical involutive isometry at the origin is the point reflection $s_0(x) = -x$, whose only fixed point is 0.

Example 2.11. Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with the standard round metric. In this case, the isometry group is

$$\text{Isom}(\mathbb{S}^2) = \{\Phi_A : \mathbb{S}^2 \rightarrow \mathbb{S}^2 \mid \Phi_A(x) = Ax, A \in \text{O}(3)\} \simeq \text{O}(3).$$

The identity component is therefore $G := \text{Isom}_0(\mathbb{S}^2) \simeq \text{SO}(3)$. It acts transitively on $\mathbb{S}^2$: given any two points on the sphere, one can rotate the first point to the second.

Choose the base point $o = e_3 = (0, 0, 1)$. The elements of G fixing o are precisely the rotations around the vertical axis. They form the stabilizer subgroup

$$K := G_o = \{A \in \text{SO}(3) : Ae_3 = e_3\} \simeq \text{SO}(2).$$

Thus, from the symmetric space point of view, the sphere can be written as

$$\mathbb{S}^2 \simeq \text{SO}(3) / \text{SO}(2).$$

The canonical involutive isometry at the base point $o = e_3$ is the rotation by π around the vertical axis: $s_o(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$.

These examples illustrate a general principle: if a group of isometries acts transitively on a space, then the space can be identified with the quotient of the group by the stabilizer of one point. Although hyperbolic space has no Euclidean type translations, it still has enough isometries to move any point to any other. In the hyperboloid model, these isometries come from linear transformations of $\mathbb{R}^{1+n}$ preserving the Lorentz form:

$$\text{SO}(1, n) = \{A \in \text{SL}(n+1, \mathbb{R}) : L(Ax, Ay) = L(x, y) \text{ for all } x, y \in \mathbb{R}^{1+n}\}.$$

Since

$$L(x, y) = x^T J_{n+1} y, \quad J_{n+1} = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix},$$

we also have

$$\mathrm{SO}(1, n) = \{A \in \mathrm{SL}(1 + n, \mathbb{R}) : A^T J_{n+1} A = J_{n+1}\}.$$

Proposition 2.12. $\mathbb{H}^n \simeq \mathrm{SO}_0(1, n)/\mathrm{SO}(n)$.

Proof. Put $G = \mathrm{SO}_0(1, n)$ and $o = (1, 0, \dots, 0) \in \mathbb{H}^n$. By definition, every element of G preserves the Lorentz form, hence preserves the equation $L(x, x) = -1$. Since G is the identity component, it also preserves the upper sheet $x_0 > 0$. Thus $g\mathbb{H}^n \subset \mathbb{H}^n$ for every $g \in G$, i.e., G acts on $\mathbb{H}^n$.

We next prove that the action is transitive, that is, for every $x \in \mathbb{H}^n$, there exists $g \in G$ such that $go = x$. We may assume that $x \neq o$. In geodesic polar coordinates, we can write

$$x = (\cosh r, \sinh r \omega), \quad r > 0, \quad \omega \in \mathbb{S}^{n-1}.$$

Choose $R \in \mathrm{SO}(n)$ such that $Re_1 = \omega$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Define the Lorentz boost a_r and the rotation k_R by

$$a_r = \begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} \quad \text{and} \quad k_R = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}.$$

Then we have

$$k_R a_r o = (\cosh r, \sinh r \omega) = x.$$

Intuitively, the Lorentz boost a_r is the hyperbolic analogue of translating the base point along the standard direction e_1 . It sends $o = (1, 0, \dots, 0)$ to $(\cosh r, \sinh r e_1)$. The rotation k_R sends e_1 to ω . Since both matrices a_r and k_R belong to G , every point of $\mathbb{H}^n$ lies in the G -orbit of o . In other words, G acts on $\mathbb{H}^n$ transitively.

We now identify the stabilizer of o . Let $K = G_o = \{g \in G : go = o\}$. We claim that

$$K = K' := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \mid R \in \mathrm{SO}(n) \right\}. \quad (2.2)$$

It is clear that all the matrices in K' belong to G and fix the base point o . Then we have $K' \subset K$. For any $g \in G$, if $go = o$, then it has to have the form

$$\begin{pmatrix} 1 & b^T \\ 0 & R \end{pmatrix}$$

where $b \in \mathbb{R}^n$ and R is an $n \times n$ matrix. Since $g \in G$ preserves the Lorentz form, we deduce from the equality $g^T J_{n+1} g = J_{n+1}$ that $b = 0$ and $R^T R = I_n$, i.e., $R \in \mathrm{O}(n)$. Moreover, since $\det(R) = \det(g) = 1$, it follows that $R \in \mathrm{SO}(n)$. We have therefore proved that $K = K' \simeq \mathrm{SO}(n)$.

We have just proved that G acts transitively on $\mathbb{H}^n$ and that the stabilizer of the base point o is K . Notice that the elements of G/K are not elements of G themselves, but cosets

$$gK = \{gk : k \in K\}.$$

Therefore, in order to obtain a concrete isomorphism, we need to specify which point of $\mathbb{H}^n$ is associated with each coset gK . This correspondence is given by the map

$$\Phi : G/K \longrightarrow \mathbb{H}^n, \quad \Phi(gK) = go.$$

Geometrically, the coset gK consists of all group elements of the form gk , $k \in K$. Since each $k \in K$ fixes the base point o , we have $(gk)o = g(ko) = go$. Thus g and gk send o to the same point. In other words, right multiplication by $K = \text{SO}(n)$ only changes the rotational freedom at the base point, it does not change the resulting point in $\mathbb{H}^n$. Hence the coset gK represents precisely the point $go \in \mathbb{H}^n$.

To make this identification rigorous, we need to prove that Φ is a one-to-one correspondence. First, Φ is well-defined. Suppose that $g_1K = g_2K$, then there exists $k \in K$ such that $g_2 = g_1k$. Since K is the stabilizer of G at the base point, we have $g_2o = (g_1k)o = g_1(ko) = g_1o$. Hence, $\Phi(g_2K) = \Phi(g_1K)$. Second, Φ is surjective, since G acts transitively on $\mathbb{H}^n$. Third, Φ is injective. Suppose that $\Phi(g_1K) = \Phi(g_2K)$. Then

$$g_1o = g_2o \iff g_2^{-1}g_1o = o \iff g_2^{-1}g_1 \in K.$$

We obtain therefore $g_1K = g_2K$. Thus Φ is one-to-one. $\square$

The discussion above identifies hyperbolic space as the homogeneous space G/K , where $G \simeq \text{SO}_0(1, n)$ and $K \simeq \text{SO}(n)$. This viewpoint explains how the isometries of $\mathbb{H}^n$ move the base point, and why quotienting by K removes exactly the symmetries that do not change that point.

2.2.2 Cartan decomposition

Already in the proof of Proposition 2.12, we saw that an isometry in G can be separated into angular parts, which lie in K , and a radial part, which is generated by a Lorentz boost. This is precisely the Cartan decomposition of the group G . It is the group-theoretic counterpart of geodesic polar coordinates on $\mathbb{H}^n$, and it explains why the middle Cartan term measures the Riemannian distance from the base point.

Let

$$H_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbf{0}_{n-1} \end{pmatrix},$$

where $\mathbf{0}_{n-1}$ denotes the zero matrix of size $(n-1) \times (n-1)$. With the notation introduced above, the one-parameter group generated by H_0 is precisely the family of Lorentz boosts used in the proof of Proposition 2.12:

$$\exp(rH_0) = a_r.$$

Therefore, if we denote $\mathfrak{a}^+ = \mathbb{R}_+ H_0$, then

$$A^+ := \exp(\mathfrak{a}^+) = \{a_r : r \geq 0\}.$$

The Cartan decomposition of $G = \text{SO}_0(1, n)$ may then be written as

$$G = KA^+K,$$

where K is of the form (2.2). In other words, for every element $g \in G$,

$$g = k_1 a_r k_2, \quad k_1, k_2 \in K, \quad r \geq 0.$$

The middle term $a_r = \exp(rH_0)$ is the radial part of g . Equivalently, if $x = go \in \mathbb{H}^n$,

$$g = k_1 \exp(x^+) k_2,$$

where $x^+ = rH_0 \in \mathfrak{a}^+$ for some $r \geq 0$. Since k_2 fixes o and k_1 acts by isometries fixing o , we have

$$d_{\mathbb{H}^n}(o, x) = d_{\mathbb{H}^n}(o, a_r o),$$

where $a_r o = (\cosh r, \sinh r e_1)$ is the unit-speed geodesic issued from o in the e_1 -direction, so

$$d_{\mathbb{H}^n}(o, x) = d_{\mathbb{H}^n}(o, a_r o) = r = |x^+|.$$

In other words, the Cartan middle term measures exactly the Riemannian distance from the base point.

Exercise 3

Define the set of matrices $\mathfrak{p} = \{X_v : v \in \mathbb{R}^n\}$, with

$$X_v = \begin{pmatrix} 0 & v^T \\ v & \mathbf{0}_n \end{pmatrix}.$$

1. Show that $\mathfrak{p} \simeq T_o \mathbb{H}^n$.
2. Prove that, for any $v, w \in \mathbb{R}^n$,

$$\langle X_v, X_w \rangle_{\mathfrak{p}} = \frac{1}{2} \operatorname{tr}(X_v X_w).$$

3. Show that $H_0 \in \mathfrak{p}$, and deduce that $|H_0| = 1$.

Remark. From the Lie algebra viewpoint, this space $\mathfrak{p}$ is the non-compact part in the Cartan decomposition $\mathfrak{so}(1, n) = \mathfrak{k} \oplus \mathfrak{p}$.

Solution to Exercise 3

1. We start by computing the tangent space at o . Since $\mathbb{H}^n$ is given by the level set $L(x, x) = -1$, a vector $\xi = (\xi_0, \xi') \in \mathbb{R}^{1+n}$ is tangent to $\mathbb{H}^n$ at o if and only if

$$0 = \left. \frac{d}{ds} \right|_{s=0} L(o + s\xi, o + s\xi) = 2L(o, \xi).$$

Since $o = (1, 0, \dots, 0)$, this condition becomes $0 = L(o, \xi) = -\xi_0$. Hence

$$T_o \mathbb{H}^n = \{(0, v) : v \in \mathbb{R}^n\}.$$

For $v \in \mathbb{R}^n$, multiplying X_v by the base point $o = (1, 0, \dots, 0)$ gives

$$X_v o = (0, v).$$

Thus the map

$$\begin{aligned}\mathfrak{p} &\longrightarrow T_o\mathbb{H}^n, \\ X_v &\longmapsto X_v o = (0, v)\end{aligned}$$

identifies the matrix X_v with the tangent vector $(0, v)$. Therefore $\mathfrak{p} \simeq T_o\mathbb{H}^n$ through this one-to-one correspondence.

2. At the base point, the hyperbolic metric is the restriction of the Lorentz form to $T_o\mathbb{H}^n$. Therefore, for $v, w \in \mathbb{R}^n$,

$$\langle X_v, X_w \rangle_{\mathfrak{p}} = L((0, v), (0, w)) = v \cdot w.$$

On the other hand,

$$X_v X_w = \begin{pmatrix} v \cdot w & 0 \\ 0 & vw^T \end{pmatrix}.$$

Thus

$$\text{tr}(X_v X_w) = v \cdot w + \text{tr}(vw^T) = 2v \cdot w,$$

from which the desired equality follows. This shows that the trace-type inner product on $\mathfrak{p}$ agrees with the Riemannian inner product at the base point under the identification $X_v \leftrightarrow (0, v)$.

3. Note that, if $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$, then

$$H_0 = X_{e_1}.$$

Hence $H_0 \in \mathfrak{p}$. Moreover,

$$\langle H_0, H_0 \rangle_{\mathfrak{p}} = \frac{1}{2} \text{tr}(H_0^2) = e_1 \cdot e_1 = 1.$$

Therefore $|H_0| = 1$.

2.2.3 Radial functions on hyperbolic space

We say that a function f on $\mathbb{H}^n$ is radial with respect to the base point o if it depends only on the Riemannian distance from o . Equivalently, there exists a function F on $[0, \infty)$ such that

$$f(x) = F(r), \quad r = d_{\mathbb{H}^n}(o, x).$$

Since K is the stabilizer of o , the elements of K are precisely the angular symmetries around the base point. Hence radially can also be described as K -invariance:

$$f(kx) = f(x), \quad k \in K, \quad x \in \mathbb{H}^n.$$

This interpretation is especially natural from the homogeneous space point of view. If f is a function on $\mathbb{H}^n \simeq G/K$, we lift it to a function $\tilde{f}$ on G by

$$\tilde{f}(g) = f(g o).$$

This lifted function is automatically right K -invariant, because every element of K fixes o . Moreover, f is radial if and only if $\tilde{f}$ is also left K -invariant. Thus radial functions on $\mathbb{H}^n$ correspond to bi- K -invariant functions on G .

2.2.4 Haar measure and the measure on hyperbolic space

Since G is a Lie group, it is a finite-dimensional smooth manifold and hence locally compact. Every locally compact group carries a left-invariant measure, called a Haar measure, which we denote by dg . Moreover, $G = \mathrm{SO}_0(1, n)$ is semisimple, and hence unimodular.³ Here unimodularity means that the Haar measure is both left and right invariant:

$$\int_G \tilde{f}(g_1 g g_2) dg = \int_G \tilde{f}(g) dg,$$

for every $g_1, g_2 \in G$. Since K is compact, it also has a Haar measure. We normalize it by

$$\int_K dk = 1.$$

This normalization removes the irrelevant volume factor coming from the angular symmetries that fix the base point.

The quotient space G/K then carries a G -invariant quotient measure induced by the Haar measure on G and the normalized Haar measure on K . We denote it by $d(gK)$. It is characterized by the formula

$$\int_G \tilde{f}(g) dg = \int_{G/K} \left(\int_K \tilde{f}(gk) dk \right) d(gK),$$

for suitable functions $\tilde{f}$ on G . This formula says that integration on G can be decomposed into integration first along the fibers gK , and then over the quotient space G/K .

Under the one-to-one identification between G/K and $\mathbb{H}^n$, the quotient measure $d(gK)$ corresponds to the Riemannian measure $d\mu_{\mathbb{H}^n}$ on hyperbolic space, after choosing the normalization accordingly. Thus, for $f \in C_c(\mathbb{H}^n)$, we may write

$$\int_{\mathbb{H}^n} f(x) d\mu_{\mathbb{H}^n}(x) = \int_{G/K} f(g o) d(gK).$$

Equivalently, if $\tilde{f}(g) = f(g o)$ is the lift of f to G , then

$$\int_{\mathbb{H}^n} f(x) d\mu_{\mathbb{H}^n}(x) = \int_G \tilde{f}(g) dg. \tag{2.3}$$

³We do not develop the Lie algebra theory here. Recall only that a connected Lie group G is called semisimple if its Lie algebra $\mathfrak{g}$ is semisimple, which means that $\mathfrak{g}$ has no nonzero solvable ideals. Equivalently, the Killing form $B(X, Y) = \mathrm{tr}(\mathrm{ad} X \mathrm{ad} Y)$ is nondegenerate. For a semisimple Lie algebra one has $\mathrm{tr}(\mathrm{ad} X) = 0$ for all $X \in \mathfrak{g}$, where $\mathrm{ad} X(Y) = [X, Y]$. For a connected Lie group, this condition is equivalent to unimodularity.

Indeed, $\tilde{f}$ is right K -invariant, and the normalization $\int_K dk = 1$ removes the integration over the K -fiber.

Using geodesic polar coordinates, we obtain the following integration formula.

Proposition 2.13. *For every $f \in C_c(\mathbb{H}^n)$,*

$$\int_{\mathbb{H}^n} f(x) d\mu_{\mathbb{H}^n}(x) = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\cosh r, \sinh r \omega) (\sinh r)^{n-1} d\omega dr.$$

In particular, if f is radial, so that $f(x) = F(r)$, then

$$\int_{\mathbb{H}^n} f(x) d\mu_{\mathbb{H}^n}(x) = |\mathbb{S}^{n-1}| \int_0^\infty F(r) (\sinh r)^{n-1} dr.$$

Corollary 2.14. *For any $\tilde{f} \in C_c(G)$,*

$$\int_G \tilde{f}(g) dg = |\mathbb{S}^{n-1}| \int_K \int_0^\infty \int_K \tilde{f}(k_1 a_r k_2) (\sinh r)^{n-1} dk_2 dr dk_1.$$

In particular, if $\tilde{f}$ is a bi- K -invariant function on G ,

$$\int_G \tilde{f}(g) dg = |\mathbb{S}^{n-1}| \int_0^\infty \tilde{f}(a_r) (\sinh r)^{n-1} dr.$$

Proof. For each $x \in \mathbb{H}^n$, choose $g_x \in G$ such that $g_x o = x$, and define

$$f(x) := \int_K \tilde{f}(g_x k_2) dk_2.$$

Note that, if $g'_x o = x$, then $g'_x = g_x k_0$ for some $k_0 \in K$, and the invariance of the Haar measure on K gives

$$\int_K \tilde{f}(g'_x k_2) dk_2 = \int_K \tilde{f}(g_x k_0 k_2) dk_2 = \int_K \tilde{f}(g_x k_2) dk_2.$$

In other words, this definition is independent of the choice of g_x . Equivalently, $f(gK)$ is well defined on the quotient space G/K .

According to formula (2.3) and the previous proposition, we have

$$\begin{aligned} \int_G \tilde{f}(g) dg &= \int_{\mathbb{H}^n} f(x) d\mu_{\mathbb{H}^n}(x) \\ &= \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\cosh r, \sinh r \omega) (\sinh r)^{n-1} d\omega dr. \end{aligned}$$

For any continuous function h on $\mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} h(\omega) d\omega = \int_{\mathbb{S}^{n-1}} \int_K h(k\omega) dk d\omega.$$

Choose $q \in K$ such that $\omega = qe_1$. Then, by the right invariance of Haar measure on K ,

$$\int_K h(k\omega) dk = \int_K h(kqe_1) dk = \int_K h(ke_1) dk.$$

Therefore,

$$\int_{\mathbb{S}^{n-1}} h(\omega) d\omega = |\mathbb{S}^{n-1}| \int_K h(ke_1) dk. \quad (2.4)$$

We deduce from (2.4) that

$$\begin{aligned} \int_G \tilde{f}(g) dg &= |\mathbb{S}^{n-1}| \int_0^\infty \int_K f(\cosh r, \sinh r k_1 e_1) (\sinh r)^{n-1} dk_1 dr \\ &= |\mathbb{S}^{n-1}| \int_0^\infty \int_K f(k_1 a_r o) (\sinh r)^{n-1} dk_1 dr \\ &= |\mathbb{S}^{n-1}| \int_0^\infty \int_K \int_K \tilde{f}(k_1 a_r k_2) dk_2 dk_1 dr. \end{aligned}$$

Since $\tilde{f} \in C_c(G)$, all the preceding integrals are absolutely convergent, and Fubini's theorem allows us to interchange their order. This completes the proof. $\square$

2.3 Spherical Fourier analysis on hyperbolic spaces

In this subsection, we develop the Fourier analysis adapted to the geometry of $\mathbb{H}^n$. In Euclidean space, the Fourier transform is built from plane waves indexed by frequency vectors. On hyperbolic space, their role is played by Helgason plane waves, which are indexed by a spectral parameter and a direction at infinity. We first explain this construction and introduce the Helgason–Fourier transform. For radial functions, averaging over the boundary direction leads naturally to spherical functions and the spherical Fourier transform. We then discuss the inversion and Plancherel formulas, the Harish–Chandra $\mathbf{c}$ -function, and the estimates and convolution properties needed later in the study of the Schrödinger equation.

2.3.1 Helgason plane waves

Recall that the Euclidean Fourier transform is defined by

$$\mathcal{F}_{\mathbb{R}^n} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx.$$

Writing $\xi = \lambda\omega$, with $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{S}^{n-1}$, the Fourier kernel $e^{-i\lambda x \cdot \omega}$ is the plane wave, which encodes the position of x along the direction ω , and the rate of oscillation λ in that direction. Note that

$$|\nabla(x \cdot \omega)| = 1 \quad \text{and} \quad \Delta(x \cdot \omega) = 0.$$

Then, this plane wave is a generalized eigenfunction of the Euclidean Laplacian:

$$\Delta_{\mathbb{R}^n} e^{-i\lambda x \cdot \omega} = -\lambda^2 e^{-i\lambda x \cdot \omega}.$$

In hyperbolic spaces, we no longer have the linear coordinate $x \cdot \omega$ available in the Euclidean setting. However, the essential ingredient of a plane wave is not the linearity of this coordinate, but the existence of a function measuring the position of a point along a given direction. This suggests replacing the Euclidean linear function $x \cdot \omega$ by a geometric analogue adapted to hyperbolic spaces. Such a role is played by the Busemann function associated with a point at infinity.

Ideal boundary. The ideal boundary $\partial_\infty \mathbb{H}^n$ may be described in terms of geodesic rays. Let $\gamma_1, \gamma_2 : [0, \infty) \rightarrow \mathbb{H}^n$ be unit-speed geodesic rays. We say that γ_1 and γ_2 are asymptotic, and write $\gamma_1 \sim \gamma_2$, if

$$\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

Then

$$\partial_\infty \mathbb{H}^n = \{\text{unit-speed geodesic rays in } \mathbb{H}^n\} / \sim.$$

Here the quotient by $\sim$ means that each boundary point is an equivalence class of geodesic rays. More precisely, for a unit-speed geodesic ray γ , we write

$$[\gamma] = \{\eta : \eta \text{ is a unit-speed geodesic ray and } \eta \sim \gamma\}.$$

The equivalence class $[\gamma]$ is one point of $\partial_\infty \mathbb{H}^n$. In other words, a point at infinity is not a point inside $\mathbb{H}^n$, but an asymptotic direction, and all geodesic rays that remain at bounded distance from one another determine the same boundary point. Thus, when we write $\gamma(t) \rightarrow b \in \partial_\infty \mathbb{H}^n$, we mean that the geodesic ray γ represents the asymptotic class b .

The ideal boundary $\partial_\infty \mathbb{H}^n$ can be described with the homogeneous space notation. Recall that $K \simeq \text{SO}(n)$ acts on the set of directions $\mathbb{S}^{n-1}$. Fix $e_1 \in \mathbb{S}^{n-1}$ and define

$$M = \{k \in K : ke_1 = e_1\} \simeq \text{SO}(n-1).$$

Thus M is the subgroup of rotations that do not change the direction e_1 . Therefore two elements $k_1, k_2 \in K$ determine the same direction if and only if $k_1 e_1 = k_2 e_1$, equivalently $k_2^{-1} k_1 \in M$. In other words, the relevant object is not the individual element $k \in K$, but the coset kM . The map

$$\begin{aligned} K/M &\longrightarrow \mathbb{S}^{n-1}, \\ kM &\longmapsto ke_1 \end{aligned}$$

is well-defined and bijective: it is well-defined because elements of M fix e_1 , surjective because K acts transitively on $\mathbb{S}^{n-1}$, and injective because two elements giving the same direction differ by an element of M . The notation Ke_1 denotes the orbit of e_1 under this action. Hence

$$K/M \simeq \{ke_1 : k \in K\} = Ke_1 = \mathbb{S}^{n-1},$$

and we may identify

$$\partial_\infty \mathbb{H}^n \simeq K/M \simeq \mathbb{S}^{n-1}.$$

Busemann function. For a fixed direction $b \in \partial_\infty \mathbb{H}^n$, let $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$ be a unit-speed geodesic such that $\gamma(0) = o$ and $\gamma(t) \rightarrow b$. A Busemann function is defined by

$$\beta_b(x) = \lim_{t \rightarrow \infty} (d(x, \gamma(t)) - t).$$

Remark 2.15. In $\mathbb{R}^n$, take $\gamma(t) = t\omega$, with a fixed direction $\omega \in \mathbb{S}^{n-1}$. Then

$$d(x, \gamma(t)) = |x - t\omega| = \sqrt{t^2 - 2t(x \cdot \omega) + |x|^2} = t - x \cdot \omega + o(1),$$

from which we deduce that $\beta_b(x) = -x \cdot \omega$.

The following identities show that the Busemann function plays the role of a geometric coordinate along the direction b . The Laplacian identity reflects the special geometry of hyperbolic space, namely the constant mean curvature of its horospheres.

Proposition 2.16. *Let $b \in \mathbb{S}^{n-1}$, and consider the unit-speed geodesic ray $\gamma_b(t) = (\cosh t, \sinh t b)$, with $t \geq 0$. The associated Busemann function is*

$$\beta_b(x) = \log(x_0 - x' \cdot b), \quad x = (x_0, x') \in \mathbb{H}^n.$$

Moreover,

$$|\nabla_{\mathbb{H}^n} \beta_b| = 1 \quad \text{and} \quad \Delta_{\mathbb{H}^n} \beta_b = n - 1. \quad (2.5)$$

Proof. We verify both identities in the hyperboloid model. By the distance formula,

$$\cosh d_{\mathbb{H}^n}(x, \gamma_b(t)) = -L(x, \gamma_b(t)).$$

Consequently,

$$\begin{aligned} \cosh d_{\mathbb{H}^n}(x, \gamma_b(t)) &= x_0 \cosh t - x' \cdot b \sinh t \\ &= \frac{e^t}{2}(x_0 - x' \cdot b) + \frac{e^{-t}}{2}(x_0 + x' \cdot b). \end{aligned}$$

Since $x_0 > |x'|$ and $|b| = 1$,

$$x_0 - x' \cdot b \geq x_0 - |x'| > 0.$$

Using

$$\operatorname{arcosh} R = \log(2R) + o(1) \quad (R \rightarrow \infty),$$

we obtain

$$d_{\mathbb{H}^n}(x, \gamma_b(t)) = t + \log(x_0 - x' \cdot b) + o(1).$$

It follows from the definition of the Busemann function that

$$\beta_b(x) = \lim_{t \rightarrow \infty} (d_{\mathbb{H}^n}(x, \gamma_b(t)) - t) = \log(x_0 - x' \cdot b).$$

We now calculate its gradient and Laplacian. In geodesic polar coordinates in the hyperboloid model,

$$x = (\cosh r, \sinh r \omega), \quad \omega \in \mathbb{S}^{n-1}.$$

Set

$$u(r, \omega) = x_0 - x' \cdot b = \cosh r - \sinh r (\omega \cdot b).$$

Thus $\beta_b = \log u$. We shall use the elementary identities

$$|\nabla_{\mathbb{S}^{n-1}}(\omega \cdot b)|^2 = 1 - (\omega \cdot b)^2, \quad \Delta_{\mathbb{S}^{n-1}}(\omega \cdot b) = -(n-1)(\omega \cdot b). \quad (2.6)$$

Indeed, the spherical gradient of $\omega \mapsto \omega \cdot b$ is the tangential projection of b :

$$\nabla_{\mathbb{S}^{n-1}}(\omega \cdot b) = b - (\omega \cdot b)\omega.$$

Hence

$$|\nabla_{\mathbb{S}^{n-1}}(\omega \cdot b)|^2 = |b - (\omega \cdot b)\omega|^2 = 1 - (\omega \cdot b)^2.$$

The Laplacian identity follows from the Euclidean polar coordinates:

$$\begin{aligned}
0 &= \Delta_{\mathbb{R}^n}(R(\omega \cdot b)) \\
&= \partial_R^2(R(\omega \cdot b)) + \frac{n-1}{R} \partial_R(R(\omega \cdot b)) + \frac{1}{R^2} \Delta_{\mathbb{S}^{n-1}}(R(\omega \cdot b)) \\
&= \frac{1}{R} ((n-1)(\omega \cdot b) + \Delta_{\mathbb{S}^{n-1}}(\omega \cdot b)),
\end{aligned}$$

which proves (2.6).

The hyperbolic metric in these coordinates is $ds^2 = dr^2 + \sinh^2 r d\omega^2$. Therefore, for any smooth function $u = u(r, \omega)$,

$$|\nabla_{\mathbb{H}^n} u|^2 = |\partial_r u|^2 + \frac{1}{\sinh^2 r} |\nabla_{\mathbb{S}^{n-1}} u|^2.$$

For $u(r, \omega) = \cosh r - \sinh r (\omega \cdot b)$, we have

$$\partial_r u = \sinh r - \cosh r (\omega \cdot b) \quad \text{and} \quad \nabla_{\mathbb{S}^{n-1}} u = -\sinh r \nabla_{\mathbb{S}^{n-1}}(\omega \cdot b).$$

Consequently,

$$\begin{aligned}
|\nabla_{\mathbb{H}^n} u|^2 &= (\sinh r - \cosh r (\omega \cdot b))^2 + |\nabla_{\mathbb{S}^{n-1}}(\omega \cdot b)|^2 \\
&= (\sinh r - \cosh r (\omega \cdot b))^2 + 1 - (\omega \cdot b)^2 \\
&= (\cosh r - \sinh r (\omega \cdot b))^2 \\
&= u^2.
\end{aligned}$$

Since $\beta_b = \log u$,

$$|\nabla_{\mathbb{H}^n} \beta_b|^2 = \frac{|\nabla_{\mathbb{H}^n} u|^2}{u^2} = 1.$$

It remains to compute the Laplacian. Recall that

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1) \coth r \partial_r + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{n-1}}.$$

We have

$$\partial_r^2 u = u \quad \text{and} \quad \Delta_{\mathbb{S}^{n-1}} u = -\sinh r \Delta_{\mathbb{S}^{n-1}}(\omega \cdot b) = (n-1) \sinh r (\omega \cdot b).$$

It follows that

$$\begin{aligned}
\Delta_{\mathbb{H}^n} u &= u + (n-1) \coth r (\sinh r - \cosh r (\omega \cdot b)) + (n-1) \frac{\omega \cdot b}{\sinh r} \\
&= u + (n-1) \left(\cosh r - \frac{\cosh^2 r}{\sinh r} (\omega \cdot b) + \frac{\omega \cdot b}{\sinh r} \right) \\
&= u + (n-1) (\cosh r - \sinh r (\omega \cdot b)) \\
&= nu.
\end{aligned}$$

Finally, the identity

$$\Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}$$

gives

$$\Delta_{\mathbb{H}^n} \beta_b = n - 1.$$

The calculation was performed for $r > 0$, where geodesic polar coordinates are regular. Since $u > 0$ and $\beta_b = \log u$ are smooth on all of $\mathbb{H}^n$, the identities extend to $r = 0$ by continuity. $\square$

In the rest of this section, we will write $\rho = (n-1)/2$. According to (2.5), we have, for any complex parameter z ,

$$\Delta_{\mathbb{H}^n} e^{z\beta_b} = (z^2 + 2\rho z) e^{z\beta_b} = ((z + \rho)^2 - \rho^2) e^{z\beta_b},$$

where $z + \rho$ is expected to be a purely imaginary number. Taking $z + \rho = -i\lambda$ with $\lambda \in \mathbb{R}$, we obtain

$$\Delta_{\mathbb{H}^n} e^{-(i\lambda+\rho)\beta_b} = -(\lambda^2 + \rho^2) e^{-(i\lambda+\rho)\beta_b}.$$

2.3.2 Spherical Fourier transform

We now use the Busemann function introduced above to define the Fourier transform on hyperbolic space. For $b \in \mathbb{S}^{n-1}$ and $x = (x_0, x') \in \mathbb{H}^n$, we denote by

$$e_{\lambda,b}(x) := e^{-(i\lambda+\rho)\beta_b(x)} = (x_0 - x' \cdot b)^{-i\lambda-\rho}$$

the so-called Helgason plane wave.

Definition 2.17 (Helgason–Fourier transform). *For $f \in C_c^\infty(\mathbb{H}^n)$, its Helgason–Fourier transform is defined by*

$$\mathcal{F}f(\lambda, b) = \int_{\mathbb{H}^n} f(x) e_{\lambda,b}(x) d\mu_{\mathbb{H}^n}(x) \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{S}^{n-1}.$$

Corollary 2.18 (Harish–Chandra transform). *Let $f \in C_c^\infty(\mathbb{H}^n)$ be a radial function: $f(x) = F(r)$ where $r = d(o, x)$. The spherical Fourier transform is*

$$\mathcal{H}f(\lambda) = \int_{\mathbb{H}^n} f(x) \varphi_\lambda(x) d\mu_{\mathbb{H}^n}(x) \tag{2.7}$$

$$= |\mathbb{S}^{n-1}| \int_0^\infty F(r) \varphi_\lambda(r) (\sinh r)^{n-1} dr, \quad \lambda \in \mathbb{R}, \tag{2.8}$$

where $\varphi_\lambda(x) = \varphi_\lambda(r)$ is the elementary spherical function satisfying $\varphi_\lambda(0) = 1$ and the explicit expression

$$\varphi_\lambda(r) = \frac{2^{\frac{n-3}{2}} \Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} (\sinh r)^{2-n} \int_{-r}^r (\cosh r - \cosh s)^{\frac{n-3}{2}} e^{-i\lambda s} ds, \tag{2.9}$$

for all $r > 0$.

Proof. According to Proposition 2.13, we have

$$\int_{\mathbb{H}^n} f(x) e_{\lambda,b}(x) d\mu_{\mathbb{H}^n}(x) = |\mathbb{S}^{n-1}| \int_0^\infty \int_K f(ka_r o) e_{\lambda,b}(ka_r o) (\sinh r)^{n-1} dk dr,$$

where, in the geodesic polar coordinates, $ka_r o = (\cosh r, \sinh r ke_1)$. As f is radial, we have

$$\mathcal{F}f(\lambda, b) = |\mathbb{S}^{n-1}| \int_0^\infty F(r) (\sinh r)^{n-1} \left(\int_K e_{\lambda,b}(ka_r o) dk \right) dr.$$

Since K acts transitively on $\mathbb{S}^{n-1}$, we may write $b = qe_1$ for some $q \in K$ and $e_1 \in \mathbb{S}^{n-1}$. Then

$$\begin{aligned} \int_K e_{\lambda,b}(ka_r o) dk &= \int_K (\cosh r - \sinh r (ke_1) \cdot b)^{-i\lambda-\rho} dk \\ &= \int_K (\cosh r - \sinh r (ke_1) \cdot (qe_1))^{-i\lambda-\rho} dk \\ &= \int_K (\cosh r - \sinh r (q^{-1}ke_1) \cdot e_1)^{-i\lambda-\rho} dk \\ &= \int_K (\cosh r - \sinh r (k'e_1) \cdot e_1)^{-i\lambda-\rho} dk', \end{aligned}$$

which is independent of b , and we denote it by $\varphi_\lambda(r)$. Choosing $b = e_1$ gives, for any $r > 0$,

$$\begin{aligned} \varphi_\lambda(r) &= \int_K e_{\lambda,e_1}(ka_r o) dk \\ &= \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} (\cosh r - \sinh r \omega \cdot e_1)^{-i\lambda-\rho} d\omega. \end{aligned}$$

Here, we have used the equality (2.4). Since the integrand depends on ω only through its angle with the fixed direction e_1 , writing $t = \omega \cdot e_1$ reduces the spherical integral to one dimension, and the standard spherical-coordinate formula gives, for any suitable function h ,

$$\int_{\mathbb{S}^{n-1}} h(\omega \cdot e_1) d\omega = |\mathbb{S}^{n-2}| \int_{-1}^1 h(t) (1-t^2)^{\frac{n-3}{2}} dt.$$

Therefore

$$\varphi_\lambda(r) = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \int_{-1}^1 (\cosh r - \sinh r t)^{-i\lambda-\rho} (1-t^2)^{\frac{n-3}{2}} dt.$$

The corollary follows from the change of variable $e^s = \cosh r - \sinh r t$. $\square$

2.3.3 Inverse formula

The Fourier transform is useful only if it can be inverted. In the Euclidean case, the inverse formula reconstructs a function by superposing plane waves over all frequencies. On hyperbolic space the role of plane waves is played by the Helgason plane waves, and for radial functions this reconstruction reduces to an integral involving the spherical functions. To make the structure as transparent as possible, we first discuss the three-dimensional case.

Example 2.19 (Inverse formula on $\mathbb{H}^3$). In this case, the spherical function has the explicit expression

$$\varphi_\lambda(r) = \frac{1}{2 \sinh r} \int_{-r}^r e^{-i\lambda s} ds = \frac{\sin(\lambda r)}{\lambda \sinh r}. \quad (2.10)$$

The spherical Fourier transform (??) becomes

$$\begin{aligned} \mathcal{H}f(\lambda) &= |\mathbb{S}^2| \int_0^\infty F(r) \varphi_\lambda(r) (\sinh r)^2 dr \\ &= \frac{4\pi}{\lambda} \int_0^\infty F(r) \sinh r \sin(\lambda r) dr, \end{aligned}$$

which transforms $\lambda/(4\pi)\mathcal{H}f(\lambda)$ into the classical sine transform of the function $r \mapsto F(r) \sinh r$. Therefore, we obtain

$$F(r) \sinh r = \frac{1}{2\pi^2} \int_0^\infty \mathcal{H}f(\lambda) \lambda \sin(\lambda r) d\lambda,$$

which gives, using (2.10) once again,

$$F(r) = \frac{1}{2\pi^2} \int_0^\infty \mathcal{H}f(\lambda) \varphi_\lambda(r) \lambda^2 d\lambda.$$

Remark 2.20. More generally, the spectral measure on $\mathbb{H}^n$ is, up to a normalization constant,

$$|\mathbf{c}(\lambda)|^{-2} d\lambda,$$

where $\mathbf{c}(\lambda)$ is the Harish–Chandra $\mathbf{c}$ -function. It satisfies

$$|\mathbf{c}(\lambda)|^{-2} \asymp \begin{cases} |\lambda|^2, & |\lambda| \ll 1, \\ |\lambda|^{n-1}, & |\lambda| \gg 1. \end{cases}$$

One way to understand its role is to look at a spherical function far from the origin. Its leading behavior is

$$\varphi_\lambda(r) \sim \mathbf{c}(\lambda) e^{(i\lambda - \rho)r} + \mathbf{c}(-\lambda) e^{(-i\lambda - \rho)r}, \quad r \rightarrow \infty.$$

Thus $\mathbf{c}(\lambda)$ measures the asymptotic amplitude of the generalized eigenfunction at frequency λ . Intuitively, on the real line,

$$\int_{\mathbb{R}} e^{i\lambda r} e^{-i\mu r} dr = 2\pi \delta(\lambda - \mu).$$

If the two oscillating waves carry the amplitudes $\mathbf{c}(\lambda)$ and $\overline{\mathbf{c}(\mu)}$, respectively, then the coefficient of the delta distribution becomes

$$\mathbf{c}(\lambda) \overline{\mathbf{c}(\mu)} \delta(\lambda - \mu) = |\mathbf{c}(\lambda)|^2 \delta(\lambda - \mu).$$

Therefore the spectral measure must contain $|\mathbf{c}(\lambda)|^{-2} d\lambda$ in order to recover the standard delta normalization and hence the identity operator.

Theorem 2.21. Let $f \in C_c^\infty(\mathbb{H}^n)$ be radial, and write $f(x) = F(r)$ with $r = d_{\mathbb{H}^n}(o, x)$. Let $\mathcal{H}f(\lambda)$ be its spherical Fourier transform. Then

$$F(r) = \frac{2^{n-3} \Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}} \int_0^\infty \mathcal{H}f(\lambda) \varphi_\lambda(r) |\mathbf{c}(\lambda)|^{-2} d\lambda.$$

2.3.4 Basic properties and estimates

We collect in this section some properties of the spherical Fourier transform that will be used later. They show that spherical functions decay in the spatial variable, the Plancherel density has different behavior at low and high frequencies, and the spherical Fourier transform diagonalizes the Laplace–Beltrami operator. Together with the inversion formula, these properties make the spherical transform the natural Fourier transform for radial analysis on $\mathbb{H}^n$.

Proposition 2.22. *Let $\lambda \in \mathbb{R}$ and $r \geq 0$. Then*

$$\varphi_\lambda(0) = 1, \quad \varphi_{-\lambda}(r) = \varphi_\lambda(r), \quad \overline{\varphi_\lambda(r)} = \varphi_\lambda(r).$$

Moreover,

$$|\varphi_\lambda(r)| \leq \varphi_0(r), \tag{2.11}$$

where

$$\varphi_0(r) \asymp (1+r)e^{-\rho r}. \tag{2.12}$$

Proof. The first three properties are easily verified from the explicit formula (2.9). The inequality (2.11) follows from the fact that

$$|e_{\lambda,b}(x)| = |e^{-(i\lambda+\rho)\beta_b(x)}| = e^{-\rho\beta_b(x)}.$$

It remains to estimate

$$\varphi_0(r) = \frac{2^{\rho-1}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}(\sinh r)^{1-2\rho} \int_{-r}^r (\cosh r - \cosh s)^{\rho-1} ds.$$

For $0 \leq r \leq 1$, the positivity and continuity of φ_0 , together with $\varphi_0(0) = 1$, show that $\varphi_0(r)$ is bounded above and below by positive constants. We therefore assume that $r \geq 1$.

By evenness in s , it is enough to consider $0 \leq s \leq r$. In this region,

$$\cosh r - \cosh s = 2 \sinh\left(\frac{r-s}{2}\right) \sinh\left(\frac{r+s}{2}\right).$$

Using the elementary estimate

$$\sinh\left(\frac{t}{2}\right) \asymp \frac{t}{1+t} e^{\frac{t}{2}}, \quad t \geq 0,$$

we obtain, for $0 \leq s < r$,

$$\cosh r - \cosh s \asymp e^r \frac{r-s}{1+r-s} \frac{r+s}{1+r+s} \asymp e^r \frac{r-s}{1+r-s}.$$

Setting $u = r - s$, we find

$$\begin{aligned} \int_{-r}^r (\cosh r - \cosh s)^{\rho-1} ds &\asymp e^{(\rho-1)r} \int_0^r \left(\frac{u}{1+u}\right)^{\rho-1} du \\ &\asymp e^{(\rho-1)r} \left(\int_0^1 u^{\rho-1} du + \int_1^r du \right) \\ &\asymp e^{(\rho-1)r} (1+r), \end{aligned}$$

for all $r \geq 1$. We then obtain

$$\varphi_0(r) \asymp e^{(1-2\rho)r} (1+r) e^{(\rho-1)r} = (1+r) e^{-\rho r} \quad r \geq 1.$$

Together with the estimate for $0 \leq r \leq 1$, we prove (2.12). □

Theorem 2.23 (Harish–Chandra expansion). *Let $\lambda \in \mathbb{R} \setminus \{0\}$. Then,*

$$\varphi_\lambda(r) = \mathbf{c}(\lambda)e^{(i\lambda-\rho)r} + \mathbf{c}(-\lambda)e^{(-i\lambda-\rho)r} + O(e^{-(\rho+2)r})$$

as $r \rightarrow \infty$, where

$$\mathbf{c}(\lambda) = \frac{2^{n-2}\Gamma(\frac{n}{2})}{\sqrt{\pi}} \frac{\Gamma(i\lambda)}{\Gamma(\rho + i\lambda)}$$

is the Harish–Chandra $\mathbf{c}$ -function.

Proof. According to the radial expression for the Laplace–Beltrami operator, the spherical function $\varphi_\lambda(r)$ satisfies the ordinary differential equation:

$$\varphi_\lambda''(r) + 2\rho \coth r \varphi_\lambda'(r) + (\lambda^2 + \rho^2)\varphi_\lambda(r) = 0. \quad (2.13)$$

To remove the first-order term in this equation, consider

$$u_\lambda(r) = (\sinh r)^\rho \varphi_\lambda(r).$$

Differentiating this identity and substituting it into (2.13), we obtain

$$u_\lambda''(r) + \lambda^2 u_\lambda(r) = V(r)u_\lambda(r), \quad V(r) = \frac{\rho(\rho-1)}{\sinh^2 r}, \quad (2.14)$$

where

$$V(r) = O(e^{-2r}) \quad \text{as } r \rightarrow \infty.$$

Hence (2.14) is an exponentially small perturbation of the free equation $u'' + \lambda^2 u = 0$. According to Duhamel's principle, the solution to (2.14) is then given by

$$u_\lambda(r) = u_\lambda(R) \cos(\lambda(r-R)) + \frac{u_\lambda'(R)}{\lambda} \sin(\lambda(r-R)) + \frac{1}{\lambda} \int_R^r \sin(\lambda(r-s)) V(s) u_\lambda(s) ds,$$

for $r \geq R > 0$. Since $V \in L^1([R, \infty))$, Gronwall's inequality yields

$$|u_\lambda(r)| \leq \left(|u_\lambda(R)| + \frac{|u_\lambda'(R)|}{|\lambda|} \right) \exp \left(\frac{1}{|\lambda|} \int_R^r |V(s)| ds \right) \leq C_{R,\lambda}.$$

Thus u_λ is bounded on $[R, \infty)$.

Let us write

$$u_\lambda(r) = \underbrace{\frac{e^{-i\lambda r}}{2i\lambda} (i\lambda u_\lambda(r) + u_\lambda'(r))}_{:=A_\lambda(r)} e^{i\lambda r} + \underbrace{\frac{e^{i\lambda r}}{2i\lambda} (i\lambda u_\lambda(r) - u_\lambda'(r))}_{:=B_\lambda(r)} e^{-i\lambda r}. \quad (2.15)$$

Using (2.14), we differentiate the coefficients A_λ and B_λ , and obtain

$$A_\lambda'(r) = \frac{e^{-i\lambda r}}{2i\lambda} V(r) u_\lambda(r) \quad \text{and} \quad B_\lambda'(r) = -\frac{e^{i\lambda r}}{2i\lambda} V(r) u_\lambda(r).$$

Since u_λ is bounded and $V(r) = O(e^{-2r})$, A_λ' and B_λ' are integrable on $[R, \infty)$. We then denote by

$$A_\infty(\lambda) = \lim_{r \rightarrow \infty} A_\lambda(r) \quad \text{and} \quad B_\infty(\lambda) = \lim_{r \rightarrow \infty} B_\lambda(r).$$

Moreover, we know that

$$|A_\lambda(r) - A_\infty(\lambda)| + |B_\lambda(r) - B_\infty(\lambda)| \leq C_\lambda \int_r^\infty |V(s)| \, ds = O(e^{-2r}).$$

We may then rewrite (2.15) as

$$u_\lambda(r) = A_\infty(\lambda)e^{i\lambda r} + B_\infty(\lambda)e^{-i\lambda r} + O(e^{-2r}).$$

Together with the estimate

$$(\sinh r)^{-\rho} = 2^\rho e^{-\rho r} (1 + O(e^{-2r})),$$

we finally find

$$\varphi_\lambda(r) = 2^\rho A_\infty(\lambda)e^{(i\lambda-\rho)r} + 2^\rho B_\infty(\lambda)e^{(-i\lambda-\rho)r} + O(e^{-(\rho+2)r}).$$

It remains then to calculate $A_\infty(\lambda)$ and $B_\infty(\lambda)$. According to (2.9), we have the explicit expression

$$\varphi_\lambda(r) = C_n (\sinh r)^{1-2\rho} \int_{-r}^r (\cosh r - \cosh s)^{\rho-1} e^{-i\lambda s} \, ds,$$

with $C_n = 2^{\rho-1} \Gamma(n/2) / (\sqrt{\pi} \Gamma(\rho))$. Split the integral at $s = 0$. In the part $-r \leq s \leq 0$, set $u = r + s$, while in the part $0 \leq s \leq r$, set $u = r - s$. This gives

$$\varphi_\lambda(r) = e^{i\lambda r - \rho r} \int_0^r K_r(u) e^{-i\lambda u} \, du + e^{-i\lambda r - \rho r} \int_0^r K_r(u) e^{i\lambda u} \, du,$$

where

$$K_r(u) = C_n e^{\rho r} (\sinh r)^{1-2\rho} (\cosh r - \cosh(r-u))^{\rho-1}.$$

Using identities

$$\sinh r = \frac{e^r}{2} (1 - e^{-2r}),$$

and

$$\cosh r - \cosh(r-u) = \frac{e^r}{2} (1 - e^{-u})(1 - e^{-2r+u}),$$

we write $K_r(u)$ as

$$K_r(u) = C_n 2^\rho (1 - e^{-2r})^{1-2\rho} (1 - e^{-u})^{\rho-1} (1 - e^{-2r+u})^{\rho-1},$$

which converges to $C_n 2^\rho (1 - e^{-u})^{\rho-1}$, as $r \rightarrow \infty$. Therefore, we have

$$2^\rho A_\infty(\lambda) = 2^\rho C_n \int_0^\infty (1 - e^{-u})^{\rho-1} e^{-i\lambda u} \, du$$

and

$$2^\rho B_\infty(\lambda) = 2^\rho C_n \int_0^\infty (1 - e^{-u})^{\rho-1} e^{i\lambda u} \, du.$$

Substituting the variable $t = e^{-u}$ gives

$$\int_0^\infty (1 - e^{-u})^{\rho-1} e^{\mp i\lambda u} \, du = \int_0^1 (1 - t)^{\rho-1} t^{\pm i\lambda - 1} \, dt.$$

The last integral is exactly the beta function

$$B(\pm i\lambda, \rho) = \frac{\Gamma(\pm i\lambda)\Gamma(\rho)}{\Gamma(\rho \pm i\lambda)}.$$

This gives the explicit formula for the Harish–Chandra $\mathbf{c}$ -function, and the proof is therefore complete. $\square$

According to the explicit expression above of the $\mathbf{c}$ -function, we have the following product representation of the Plancherel density, and the corresponding estimates at low and high frequencies.

Proposition 2.24. *The Plancherel density has the following product representation:*

$$|\mathbf{c}(\lambda)|^{-2} = \frac{\pi}{2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2} \begin{cases} \lambda^2 \prod_{j=1}^{m-1} (\lambda^2 + j^2) & n = 2m + 1, \\ \lambda \tanh(\pi\lambda) \prod_{j=0}^{m-2} \left(\lambda^2 + \left(j + \frac{1}{2}\right)^2 \right) & n = 2m. \end{cases}$$

This yields the estimates

$$|\mathbf{c}(\lambda)|^{-2} \asymp |\lambda|^2 (1 + |\lambda|)^{n-3} \asymp \begin{cases} |\lambda|^2 & 0 < |\lambda| \leq 1, \\ |\lambda|^{n-1} & |\lambda| \geq 1. \end{cases}$$

Moreover, for every integer $k \geq 0$, one also has the symbol estimates

$$\left| \partial_\lambda^k |\mathbf{c}(\lambda)|^{-2} \right| \lesssim \begin{cases} |\lambda|^{2-k} & 0 < |\lambda| \leq 1, \\ |\lambda|^{n-1-k} & |\lambda| \geq 1. \end{cases}$$

Proof. Suppose first that $n = 2m + 1$, so that $\rho = m$. The recurrence relation for the Gamma function gives

$$\frac{\Gamma(m + i\lambda)}{\Gamma(i\lambda)} = \prod_{j=0}^{m-1} (j + i\lambda),$$

which directly yields the identity in the odd-dimensional case.

In the case where $n = 2m$, so $\rho = m - 1/2$, we have

$$\Gamma\left(m - \frac{1}{2} + i\lambda\right) = \Gamma\left(\frac{1}{2} + i\lambda\right) \prod_{j=0}^{m-2} \left(j + \frac{1}{2} + i\lambda\right).$$

The result then follows from the classical identities

$$|\Gamma(i\lambda)|^2 = \frac{\pi}{\lambda \sinh(\pi\lambda)} \quad \text{and} \quad \left| \Gamma\left(\frac{1}{2} + i\lambda\right) \right|^2 = \frac{\pi}{\cosh(\pi\lambda)}.$$

The remaining estimates follow from direct calculations using the product representation. $\square$

The following Plancherel formula for the spherical Fourier transform can be obtained through classical arguments.

Theorem 2.25. For radial functions $f, g \in C_c^\infty(\mathbb{H}^n)$,

$$\int_{\mathbb{H}^n} f(x) \overline{g(x)} d\mu_{\mathbb{H}^n}(x) = C_n \int_0^\infty \mathcal{H}f(\lambda) \overline{\mathcal{H}g(\lambda)} |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

where $C_n = 2^{n-3} \pi^{-n/2-1} \Gamma(n/2)$.

Remark 2.26. The spherical Fourier transform is an isomorphism between the following spaces:

$$\begin{aligned} C_{c,\text{rad}}^\infty(\mathbb{H}^n) & \xrightarrow[\text{topological}]{\sim} \mathcal{PW}_{\text{even}}(\mathbb{C}), \\ \mathcal{S}_{\text{rad}}(\mathbb{H}^n) & \xrightarrow[\text{topological}]{\sim} \mathcal{S}_{\text{even}}(\mathbb{R}), \\ L_{\text{rad}}^2(\mathbb{H}^n) & \xrightarrow[\text{unitary}]{\sim} L^2((0, \infty), C_n |\mathbf{c}(\lambda)|^{-2} d\lambda). \end{aligned}$$

Here $\mathcal{PW}_{\text{even}}(\mathbb{C})$ denotes the space of even entire functions F for which there exists $R > 0$ such that, for every $N \geq 0$, there is a constant $C_N > 0$ satisfying

$$|F(z)| \leq C_N (1 + |z|)^{-N} e^{R|\text{Im } z|}, \quad z \in \mathbb{C}.$$

The radial Harish–Chandra Schwartz space $\mathcal{S}_{\text{rad}}(\mathbb{H}^n)$ consists of radial smooth functions $f(x) = F(r)$, such that, for every $N, k \geq 0$,

$$\sup_{r \geq 0} (1 + r)^N \varphi_0(r)^{-1} |\partial_r^k F(r)| < \infty.$$

2.3.5 Convolution on the isometry group and the symmetric space

For $\tilde{f}_1, \tilde{f}_2 \in C_c(G)$, their convolution on G is defined by

$$(\tilde{f}_1 * \tilde{f}_2)(g) = \int_G \tilde{f}_1(h) \tilde{f}_2(h^{-1}g) dh.$$

Since G is unimodular, the same convolution may equivalently be written as

$$(\tilde{f}_1 * \tilde{f}_2)(g) = \int_G \tilde{f}_1(gh^{-1}) \tilde{f}_2(h) dh.$$

We descend convolution from G to $\mathbb{H}^n$. If k is radial about o , write $k(x) = K(d_{\mathbb{H}^n}(o, x))$. Its lift $\tilde{k}(g) = k(go)$ is bi- K -invariant. We define the convolution of f with the radial kernel k by

$$\widetilde{f * k} = \tilde{f} * \tilde{k}.$$

Equivalently, for $x \in \mathbb{H}^n$,

$$(f * k)(x) = \int_{\mathbb{H}^n} f(y) K(d_{\mathbb{H}^n}(x, y)) d\mu_{\mathbb{H}^n}(y).$$

Remark 2.27. Let f_1, f_2 be functions on $\mathbb{H}^n$, with right K -invariant lifts $\tilde{f}_1, \tilde{f}_2$ to G . Their group convolution is right K -invariant and hence descends to $G/K \simeq \mathbb{H}^n$. However,

$$\tilde{f}_1 * \tilde{f}_2 = \tilde{f}_1 * \tilde{f}_2^\#, \quad f_2^\#(x) = \int_K f_2(k^{-1}x) dk.$$

Thus the descended convolution automatically replaces the second factor by its radialization and loses its non-radial angular information. This explains why an intrinsic convolution on $\mathbb{H}^n$ is naturally formulated with one radial kernel.

The following proposition is the analogue of the classical property that the Fourier transform of a convolution is the product of the Fourier transforms. However, in the present setting, it requires one of the factors to be bi- K -invariant.

Proposition 2.28. *Let $\tilde{f}_1, \tilde{f}_2 \in C_c^\infty(G)$, and suppose that $\tilde{f}_2$ is bi- K -invariant. Then*

$$\mathcal{F}(\tilde{f}_1 * \tilde{f}_2)(\lambda, b) = \mathcal{F}\tilde{f}_1(\lambda, b) \mathcal{H}\tilde{f}_2(\lambda).$$

Proof. Using Fubini's theorem and the change of variables $g = hu$, we obtain

$$\begin{aligned} \mathcal{F}(\tilde{f}_1 * \tilde{f}_2)(\lambda, b) &= \int_G \int_G \tilde{f}_1(h) \tilde{f}_2(h^{-1}g) e_{\lambda, b}(go) \, dh \, dg \\ &= \int_G \tilde{f}_1(h) \left(\int_G \tilde{f}_2(u) e_{\lambda, b}(huo) \, du \right) dh. \end{aligned}$$

Denote by γ_b the unit-speed geodesic ray issuing from o and tending to b . Then, by definition of the Busemann function,

$$\begin{aligned} \beta_b(huo) - \beta_b(ho) &= \lim_{t \rightarrow \infty} [d_{\mathbb{H}^n}(huo, \gamma_b(t)) - d_{\mathbb{H}^n}(ho, \gamma_b(t))] \\ &= \lim_{t \rightarrow \infty} [d_{\mathbb{H}^n}(uo, h^{-1}\gamma_b(t)) - d_{\mathbb{H}^n}(o, h^{-1}\gamma_b(t))] \\ &= \lim_{z \rightarrow h^{-1}b} [d_{\mathbb{H}^n}(uo, z) - d_{\mathbb{H}^n}(o, z)] \\ &= \beta_{h^{-1}b}(uo). \end{aligned}$$

This yields the so-called Busemann cocycle property:

$$e_{\lambda, b}(huo) = e_{\lambda, b}(ho) e_{\lambda, h^{-1}b}(uo),$$

from which, we conclude that

$$\mathcal{F}(\tilde{f}_1 * \tilde{f}_2)(\lambda, b) = \int_G \tilde{f}_1(h) e_{\lambda, b}(ho) (\mathcal{F}\tilde{f}_2)(\lambda, h^{-1}b) \, dh = \mathcal{F}\tilde{f}_1(\lambda, b) \mathcal{H}\tilde{f}_2(\lambda),$$

since $\tilde{f}_2$ is bi- K -invariant. □

Remark 2.29. The left K -invariance of one factor is precisely what turns convolution into multiplication by a scalar function of λ . Without such a condition, the last equality in the proof would fail.

The following convolution estimate is a special feature of non-compact semisimple Lie groups, which improves the classical Young's inequality. It is known as the Kunze–Stein phenomenon.

Theorem 2.30. *Let G be a connected non-compact semisimple Lie group with finite center.⁴ For every $1 \leq p < 2$, there exists a constant $C_p > 0$ such that*

$$\|\tilde{f}_1 * \tilde{f}_2\|_{L^2(G)} \leq C_p \|\tilde{f}_1\|_{L^p(G)} \|\tilde{f}_2\|_{L^2(G)}$$

for all $\tilde{f}_1 \in L^p(G)$ and $\tilde{f}_2 \in L^2(G)$.

⁴The center of G is $Z(G) = \{z \in G : zg = gz, \forall g \in G\}$. Thus the elements of the center commute with every element of G . For a connected semisimple Lie group, the center is discrete but need not be finite. The finite center assumption rules out an infinite discrete central subgroup, which would introduce an abelian discrete convolution variable into G . In the present setting $G = \mathrm{SO}_0(1, n)$, one has $Z(G) = \{I\}$, so this assumption is automatically satisfied.

Remark 2.31. Young's convolution inequality, which is valid on every unimodular locally compact group, states that

$$\|\tilde{f}_1 * \tilde{f}_2\|_{L^r(G)} \leq \|\tilde{f}_1\|_{L^p(G)} \|\tilde{f}_2\|_{L^q(G)}, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

If $q = r = 2$, the relation among the exponents forces $p = 1$. Thus Young's inequality gives only

$$L^1(G) * L^2(G) \subset L^2(G),$$

whereas the Kunze–Stein inequality improves this to

$$L^p(G) * L^2(G) \subset L^2(G), \quad 1 \leq p < 2.$$

For our purposes, we only need the following version of the Kunze–Stein inequality for bi- K -invariant kernels.

Proposition 2.32. *Let $2 \leq q \leq \infty$, and let q' be its dual exponent. Then*

$$\|\tilde{f}_1 * \tilde{f}_2\|_{L^q(G)} \leq \|\tilde{f}_1\|_{L^{q'}(G)} \left(\int_G \varphi_0(g) |\tilde{f}_2(g)|^{\frac{q}{2}} dg \right)^{\frac{2}{q}} \quad (2.16)$$

for every $\tilde{f}_1 \in L^{q'}(G)$ and every bi- K -invariant function $\tilde{f}_2 \in L^{q/2}(G, \varphi_0(g) dg)$. When $q = \infty$, the last factor in (2.16) is understood as $\|\tilde{f}_2\|_{L^\infty(G)}$.

Proof. Without loss of generality, we may assume that $\tilde{f}_1$ and $\tilde{f}_2$ are nonnegative. The result follows by interpolation between the endpoints $q = \infty$ and $q = 2$. When $q = \infty$, the desired estimate is simply Young's inequality,

$$\|\tilde{f}_1 * \tilde{f}_2\|_{L^\infty(G)} \leq \|\tilde{f}_1\|_{L^1(G)} \|\tilde{f}_2\|_{L^\infty(G)}.$$

We now prove the endpoint $q = 2$. For $z \in \mathbb{C}$ with $|\operatorname{Re} z| \leq 1$, consider the analytic family

$$(T_z \tilde{f})(g) = \varphi_0(g)^{-z} [(\varphi_0^z \tilde{f}) * \tilde{f}_2](g).$$

For $\operatorname{Re} z = 1$, we have

$$|[(\varphi_0^z \tilde{f}) * \tilde{f}_2](g)| \leq \|\tilde{f}\|_{L^\infty(G)} \int_G \varphi_0(gh^{-1}) \tilde{f}_2(h) dh.$$

Since $\tilde{f}_2$ is right K -invariant, averaging over K gives

$$\int_G \varphi_0(gh^{-1}) \tilde{f}_2(h) dh = \int_G \tilde{f}_2(h) \left(\int_K \varphi_0(gkh^{-1}) dk \right) dh.$$

Because φ_0 is spherical and symmetric,

$$\int_K \varphi_0(gkh^{-1}) dk = \varphi_0(g) \varphi_0(h^{-1}) = \varphi_0(g) \varphi_0(h).$$

Consequently,

$$\|T_z \tilde{f}\|_{L^\infty(G)} \leq A \|\tilde{f}\|_{L^\infty(G)}, \quad \forall \operatorname{Re} z = 1,$$

where

$$A = \int_G \varphi_0(h) \tilde{f}_2(h) dh = \mathcal{H} \tilde{f}_2(0).$$

On the other boundary of the strip, duality gives

$$\|T_z \tilde{f}\|_{L^1(G)} \leq A \|\tilde{f}\|_{L^1(G)}, \quad \operatorname{Re} z = -1.$$

Interpolation on the strip at $z = 0$ therefore yields

$$\|\tilde{f} * \tilde{f}_2\|_{L^2(G)} = \|T_0 \tilde{f}\|_{L^2(G)} \leq A \|\tilde{f}\|_{L^2(G)}.$$

Then (2.16) follows by interpolating this L^2 estimate with the L^∞ endpoint. $\square$

III Schrödinger Equation on Hyperbolic Spaces

We now turn to the Schrödinger equation on hyperbolic space,

$$\begin{cases} i\partial_t u(t, x) + \Delta_{\mathbb{H}^n} u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{H}^n, \\ u(0, x) = u_0(x). \end{cases}$$

As in Euclidean space, the solution is generated by the free Schrödinger propagator

$$u(t) = e^{it\Delta_{\mathbb{H}^n}} u_0.$$

Let s_t denote the convolution kernel of $e^{it\Delta_{\mathbb{H}^n}}$. Thus

$$e^{it\Delta_{\mathbb{H}^n}} u_0 = u_0 * s_t. \quad (3.1)$$

Recall that the Laplace–Beltrami operator is invariant under isometries, and so is the Schrödinger propagator. Then, the kernel s_t is a radial function on $\mathbb{H}^n$. Using the Helgason–Fourier transform on the equation (3.1), and then the spherical Fourier inversion formula, we obtain the following integral representation of the kernel

$$s_t(r) = \frac{2^{n-3} \Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}} \int_0^\infty e^{-it(\lambda^2 + \rho^2)} \varphi_\lambda(r) |\mathbf{c}(\lambda)|^{-2} d\lambda. \quad (3.2)$$

3.1 Pointwise estimates for the Schrödinger kernel

The behavior of the Schrödinger kernel is different at small and large times. At small times, the local geometry of $\mathbb{H}^n$ is approximately Euclidean, and the kernel decays like $|t|^{-n/2}$. At large times, the bottom of the spectrum and the geometry at infinity become important, and the time decay is $|t|^{-3/2}$, independently of the dimension n .

Proposition 3.1 (Large-time estimate). *Let $N > \max\{n, 3\}$. For every $|t| \geq 1$ and $r \geq 0$,*

$$|s_t(r)| \lesssim |t|^{-\frac{3}{2}} \left(1 + \frac{r}{\sqrt{|t|}}\right)^N \varphi_0(r). \quad (3.3)$$

Proof. Since $s_{-t} = \overline{s_t}$, it is enough to consider $t \geq 1$. In what follows, the spectral integral is understood formally as an oscillatory integral, and all integrations by parts are performed in this sense. We shall use the integral representation of the spherical function φ_λ in terms of the Busemann function:

$$\varphi_\lambda(x) = \int_K e^{-(i\lambda + \rho)\beta_b(kx)} dk. \quad (3.4)$$

Recall that the Busemann function is 1-Lipschitz and satisfies $\beta_b(o) = 0$. Since k fixes o ,

$$|\beta_b(kx)| \leq d_{\mathbb{H}^n}(o, kx) = d_{\mathbb{H}^n}(o, x) = r.$$

Moreover, φ_λ is radial, and satisfies the identity

$$\varphi_0(r) = \int_K e^{-\rho\beta_b(kx)} dk. \quad (3.5)$$

Using (3.4), we write the expression (3.2) as

$$s_t(r) = e^{-it\rho^2} \underbrace{\frac{2^{n-3}\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}}}_{:=c_n} \int_K e^{-\rho\beta_b(kx)} \underbrace{\left\{ \int_0^\infty e^{-it\lambda^2 - i\lambda\beta_b(kx)} |\mathbf{c}(\lambda)|^{-2} d\lambda \right\}}_{:=I(t, kx)} dk \quad (3.6)$$

Thus the large-time estimate has been reduced to a one-dimensional oscillatory integral.

Choose an even cutoff function $\chi_0 \in C_c^\infty(\mathbb{R})$ such that $\chi_0 = 1$ near the origin, and let $\chi_\infty = 1 - \chi_0$. Split $I(t, kx) = I_0(t, kx) + I_\infty(t, kx)$ by inserting respectively $\chi_0(\sqrt{t}\lambda)$ and $\chi_\infty(\sqrt{t}\lambda)$. On the one hand, in the low-frequency case where $|\lambda| \lesssim t^{-1/2}$, the estimate

$$|\mathbf{c}(\lambda)|^{-2} \lesssim |\lambda|^2, \quad |\lambda| \leq 1,$$

directly gives

$$|I_0(t, kx)| \lesssim \int_{|\lambda| \lesssim t^{-1/2}} |\lambda|^2 d\lambda \lesssim t^{-\frac{3}{2}}.$$

On the other hand, in the high-frequency case where $|\lambda| \gtrsim t^{-1/2}$, we integrate by parts $N > n$ times by using

$$e^{-it\lambda^2} = \frac{i}{2t\lambda} \partial_\lambda e^{-it\lambda^2}.$$

When $m \leq N$ of the resulting derivatives fall on $e^{-i\lambda\beta_b(kx)}$, they produce a factor $|\beta_b(kx)|^m$. In the interval $t^{-1/2} \lesssim |\lambda| \leq 1$, the symbol estimates for the Plancherel density reduce the corresponding terms to

$$t^{-N} |\beta_b(kx)|^m \int_{t^{-1/2}}^1 \lambda^{2-2N+m} d\lambda \lesssim t^{-\frac{3}{2}} \left(\frac{|\beta_b(kx)|}{\sqrt{t}} \right)^m.$$

On $|\lambda| \geq 1$, they give

$$t^{-N} |\beta_b(kx)|^m \int_1^\infty \lambda^{n-1-2N+m} d\lambda \lesssim t^{-\frac{3}{2}} \left(1 + \frac{|\beta_b(kx)|}{\sqrt{t}} \right)^N.$$

Terms in which derivatives fall on $\chi_\infty(\sqrt{t}\lambda)$ are supported on the shell $|\lambda| \asymp t^{-1/2}$, and satisfy the same estimate. Consequently,

$$|I_\infty(t, kx)| \lesssim t^{-\frac{3}{2}} \left(1 + \frac{|\beta_b(kx)|}{\sqrt{t}}\right)^N \lesssim t^{-\frac{3}{2}} \left(1 + \frac{|r|}{\sqrt{t}}\right)^N.$$

Inserting this estimate into (3.6) and using (3.5) gives (3.3). $\square$

If one repeats the preceding frequency decomposition for $0 < |t| \leq 1$, the low-frequency term satisfies the acceptable bound $|t|^{-n/2} \varphi_0(r)$. However, the high-frequency part will contain an unnecessarily large negative power of $|t|$. To get the desired small-time estimate, we instead use a parametrix argument adapted to $\mathbb{H}^n$.

Proposition 3.2 (Small-time estimate). *For every $0 < |t| \leq 1$ and $r \geq 0$,*

$$|s_t(r)| \lesssim |t|^{-\frac{n}{2}} (1+r)^\rho e^{-\rho r}.$$

Proof. We shall construct a radial parametrix. Set

$$v(t, r) = (4\pi i t)^{-\frac{n}{2}} e^{-it\rho^2} e^{i\frac{r^2}{4t}} J(r)^{-\frac{1}{2}} \quad \text{with} \quad J(r) = \left(\frac{\sinh r}{r}\right)^{n-1}.$$

We seek an expansion of the form

$$s_t(r) = v(t, r) \left\{ \sum_{m=0}^N (it)^m u_m(r) + R_N(t, r) \right\}, \quad u_0(r) = 1. \quad (3.7)$$

Using the following radial expression of the Laplace–Beltrami operator on $\mathbb{H}^n$,

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1) \coth r \partial_r,$$

we know, for radial functions, that

$$J(r)^{\frac{1}{2}} \Delta_{\mathbb{H}^n} J(r)^{-\frac{1}{2}} = \partial_r^2 + \frac{n-1}{r} \partial_r - \rho^2 + \omega(r), \quad (3.8)$$

where

$$\omega(r) = \rho(\rho-1) \left(\frac{1}{r^2} - \frac{1}{\sinh^2 r} \right)$$

is a bounded, even, and analytic function on $\mathbb{R}$.

Conjugating the Schrödinger operator by v , and using (3.8), gives

$$v^{-1}(i\partial_t + \Delta_{\mathbb{H}^n})v = i\partial_t + \partial_r^2 + \left(\frac{n-1}{r} + \frac{ir}{t}\right) \partial_r + \omega(r). \quad (3.9)$$

Denote the part independent of t by

$$\mathcal{L}_0 = \partial_r^2 + \frac{n-1}{r} \partial_r + \omega(r).$$

Applying the operator $i\partial_t + \Delta_{\mathbb{H}^n}$ to the parametrix (3.7) and using (3.9), then matching equal powers of t , we obtain the transport equations

$$(r\partial_r + m)u_m(r) = \mathcal{L}_0 u_{m-1}(r), \quad 1 \leq m \leq N, \quad (3.10)$$

and the remainder equation

$$\left[i\partial_t + \partial_r^2 + \left(\frac{n-1}{r} + \frac{ir}{t} \right) \partial_r + \omega(r) \right] R_N = -(it)^N \mathcal{L}_0 u_N. \quad (3.11)$$

Multiplying (3.10) by r^{m-1} and integrating from 0 to r , we obtain

$$u_m(r) = r^{-m} \int_0^r x^{m-1} \mathcal{L}_0 u_{m-1}(x) dx = \int_0^1 y^{m-1} \mathcal{L}_0 u_{m-1}(ry) dy.$$

Since ω is even and analytic, this formula shows inductively that every u_m is even and analytic. The same induction, using $\omega(r) = O(r^{-2})$ as $r \rightarrow \infty$, gives

$$\partial_r^k u_m(r) = O(r^{-m-k}), \quad r \rightarrow \infty. \quad (3.12)$$

It remains to control the remainder. Put

$$\tilde{R}_N(t, r) = v(t, r) R_N(t, r) \quad \text{and} \quad \tilde{u}_N(t, r) = -(it)^N v(t, r) \mathcal{L}_0 u_N(r).$$

We deduce from (3.11) and (3.7) that $\tilde{R}_N$ satisfies the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta_{\mathbb{H}^n}) \tilde{R}_N = \tilde{u}_N, \\ \lim_{t \rightarrow 0} \tilde{R}_N(t, \cdot) = 0. \end{cases}$$

By Duhamel's formula,

$$\tilde{R}_N(t) = -i \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}} \tilde{u}_N(s) ds.$$

By the definition of $\mathcal{L}_0$ and estimates (3.12), we have

$$\left| \partial_r^k \mathcal{L}_0 u_N(r) \right| \lesssim (1+r)^{-N-2-k}.$$

The contributions of the factor $v(t, r)$ come from

$$\left| \partial_r^k J(r)^{-\frac{1}{2}} \right| \lesssim (1+r)^\rho e^{-\rho r} \quad \text{and} \quad \left| \partial_r^k e^{\frac{ir^2}{4t}} \right| \lesssim |t|^{-k} (1+r)^k,$$

while the terms $(it)^N (4\pi it)^{-n/2}$ contribute $|t|^{N-n/2}$. In the case where $0 < |t| \leq 1$, the worst contribution occurring when all derivatives fall on the oscillatory factor. Therefore, for all $0 \leq k \leq \sigma$,

$$\left| \partial_r^k \tilde{u}_N(t, r) \right| \lesssim |t|^{N-k-\frac{n}{2}} (1+r)^{\rho+k-N-2} e^{-\rho r},$$

and hence, for $N > n/2 + \sigma - 1$,

$$\|\tilde{u}_N(t, \cdot)\|_{H^\sigma(\mathbb{H}^n)} \lesssim |t|^{N-\frac{n}{2}-\sigma}.$$

Using the H^σ conservation and the next Lemma 3.3, we obtain

$$\begin{aligned} |\tilde{R}_N(t, r)| &\lesssim e^{-\rho r} \|\tilde{R}_N(t, \cdot)\|_{H^\sigma(\mathbb{H}^n)} \\ &\lesssim e^{-\rho r} \int_0^{|t|} s^{N-\frac{n}{2}-\sigma} ds \\ &\lesssim |t|^{N+1-\frac{n}{2}-\sigma} e^{-\rho r}, \end{aligned}$$

provided $N > n/2 + \sigma - 1$. We conclude the proof by inserting these estimates into (3.7). $\square$

Lemma 3.3. *Let σ be an integer with $\sigma > n/2$. For every radial function $f \in H^\sigma(\mathbb{H}^n)$,*

$$|f(r)| \leq C_\sigma e^{-\rho r} \|f\|_{H^\sigma(\mathbb{H}^n)}, \quad r \geq 0.$$

Proof. For $0 \leq r \leq 1$, the global Sobolev embedding $H^\sigma(\mathbb{H}^n) \hookrightarrow L^\infty(\mathbb{H}^n)$ gives the desired estimate. Suppose therefore that $r \geq 1$. Let $\chi \in C_c^\infty(\mathbb{H}^n)$ satisfy $\chi(o) = 1$ and $\text{supp } \chi \subset B(o, 1)$. Choose a maximal family of pairwise disjoint balls

$$B_j = B(x_j, 1), \quad x_j \in S(o, r), \quad 1 \leq j \leq N_r,$$

where $S(o, r)$ is the geodesic sphere of radius r centered at o . The balls B_j lie in the annulus

$$B(o, r+1) \setminus B(o, r-1).$$

Since balls of a fixed radius have the same volume at every center, the disjointness of the B_j 's and the hyperbolic volume growth give

$$N_r \lesssim \mu_{\mathbb{H}^n}(B(o, r+1) \setminus B(o, r-1)) \asymp e^{(n-1)r}.$$

The reverse inequality follows from maximality. Thus

$$N_r \asymp e^{(n-1)r} = e^{2\rho r}. \quad (3.13)$$

For each j , choose $g_j \in G$ such that $g_j o = x_j$, and define $\chi_j(x) = \chi(g_j^{-1}x)$. Then $\chi_j(x_j) = 1$, and χ_j is supported in B_j . The local Sobolev embedding on a unit geodesic ball, whose constant is independent of the center by homogeneity, gives

$$|f(x_j)| = |\chi_j(x_j)f(x_j)| \lesssim \|\chi_j f\|_{H^\sigma(\mathbb{H}^n)}.$$

Since f is radial and $d_{\mathbb{H}^n}(o, x_j) = r$, we have $f(x_j) = f(r)$ for every j . Summing over the disjoint balls, we obtain

$$N_r |f(r)|^2 \lesssim \sum_{j=1}^{N_r} \|\chi_j f\|_{H^\sigma(\mathbb{H}^n)}^2 \lesssim \|f\|_{H^\sigma(\mathbb{H}^n)}^2.$$

We obtain the desired estimate from (3.13). $\square$

3.2 Dispersive estimates

We now combine the pointwise kernel estimates with the Kunze–Stein inequality to obtain dispersive estimates for the Schrödinger propagator on hyperbolic space.

Theorem 3.4. *Let $2 < q \leq \infty$, and let q' be its dual exponent. Then*

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L^q(\mathbb{H}^n)} \leq C_q \begin{cases} |t|^{-n(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^{q'}(\mathbb{H}^n)}, & 0 < |t| \leq 1, \\ |t|^{-\frac{3}{2}} \|f\|_{L^{q'}(\mathbb{H}^n)}, & |t| \geq 1. \end{cases}$$

At the endpoint $q = 2$, the propagator is unitary:

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L^2(\mathbb{H}^n)} = \|f\|_{L^2(\mathbb{H}^n)}. \quad (3.14)$$

Proof. By the spectral theorem, the Schrödinger propagator $e^{it\Delta_{\mathbb{H}^n}}$ is unitary. Hence (3.14) holds. The small-time argument is similar to the Euclidean case. It is sufficient to interpolate between (3.14) and the $L^1 \rightarrow L^\infty$ estimate:

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L^\infty(\mathbb{H}^n)} \lesssim \|f * s_t\|_{L^\infty(\mathbb{H}^n)} \lesssim |t|^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{H}^n)},$$

which follows from Proposition 3.2.

Now suppose that $|t| \geq 1$. Since s_t is radial on $\mathbb{H}^n$, it is bi- K -invariant on G . By the Kunze–Stein phenomenon in Theorem 2.32, which descends to $\mathbb{H}^n$, we obtain

$$\|f * s_t\|_{L^q(\mathbb{H}^n)} \leq \|f\|_{L^{q'}(\mathbb{H}^n)} \left(\int_{\mathbb{H}^n} \varphi_0(x) |s_t(x)|^{\frac{q}{2}} d\mu_{\mathbb{H}^n}(x) \right)^{\frac{2}{q}}, \quad (3.15)$$

for every $2 < q < \infty$. For $q = \infty$, the last factor is interpreted as $\|s_t\|_{L^\infty(\mathbb{H}^n)}$.

Since φ_0 and s_t are both radial, we deduce from Proposition 2.13 that

$$\int_{\mathbb{H}^n} \varphi_0(x) |s_t(x)|^{\frac{q}{2}} d\mu_{\mathbb{H}^n}(x) \asymp \int_0^\infty \varphi_0(r) |s_t(r)|^{\frac{q}{2}} (\sinh r)^{n-1} dr.$$

Using the elementary spherical function estimate (2.12), together with the large-time kernel estimate (3.3), we obtain

$$\int_{\mathbb{H}^n} \varphi_0(x) |s_t(x)|^{\frac{q}{2}} d\mu_{\mathbb{H}^n}(x) \lesssim |t|^{-\frac{3q}{4}} \int_0^\infty (1+r)^{1+\frac{(N+1)q}{2}} e^{-\rho(\frac{q}{2}-1)r} dr \lesssim |t|^{-\frac{3q}{4}},$$

provided $q > 2$ and N is sufficiently large. Substituting this estimate into (3.15) gives

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L^q(\mathbb{H}^n)} \lesssim_q |t|^{-\frac{3}{2}} \|f\|_{L^{q'}(\mathbb{H}^n)}, \quad |t| \geq 1.$$

The proof is therefore complete. □

3.3 Strichartz estimates

The dispersive estimates above can now be converted into spacetime estimates by the standard TT^* argument. The long-time decay of the dispersive estimate, together with the Kunze–Stein phenomenon, gives a larger range of admissible exponents than the Euclidean admissibility line.

A pair (p, q) is called *admissible* if $(1/p, 1/q)$ belongs to the admissible triangle

$$\left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \left(0, \frac{1}{2} \right] \times \left(0, \frac{1}{2} \right] \mid \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2} \right\} \cup \left\{ \left(0, \frac{1}{2} \right) \right\}. \quad (3.16)$$

Remark 3.5. The admissible set on hyperbolic spaces is significantly larger than the one in the Euclidean setting, which only corresponds to the lower edge of the admissible triangle (3.16), see the figure below.



Figure 2: Admissible sets on hyperbolic spaces of dimension $n \geq 3$ and $n = 2$.

The enlargement beyond the Euclidean line reflects the two time scales in the dispersive estimate: the short-time singularity requires $n(1/2 - 1/q) \leq 2/p$ as in the Euclidean setting, while the Kunze–Stein phenomenon gives the integrable long-time tail $|t - s|^{-3/2}$. We will see this point in the proof of the next theorem.

Similar to the Euclidean case, for the endpoint $(p, q) = (\infty, 2)$, the L^2 conservation law gives the estimate

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L_t^\infty L_x^2} = \|f\|_{L^2(\mathbb{H}^n)}.$$

Another endpoint $(p, q) = (2, 2n/(n-2))$ can be handled by using Keel–Tao’s dyadic decomposition argument as in the Euclidean setting. We shall not pursue this endpoint here. We will prove the following Strichartz estimates for the non-endpoint admissible pairs.

Theorem 3.6. Suppose that the pair (p, q) satisfies conditions $2 \leq p < \infty$, $2 < q < \infty$,

$$\frac{2}{p} + \frac{n}{q} \geq \frac{n}{2}, \quad (3.17)$$

and $(p, q) \neq (2, 2n/(n-2))$. Then, the following Strichartz estimate holds for all $f \in L^2(\mathbb{H}^n)$:

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)} \lesssim_{p,q} \|f\|_{L^2(\mathbb{H}^n)}. \quad (3.18)$$

Proof. For suitable functions f and F , we define the operator

$$Tf(t, x) = e^{it\Delta_{\mathbb{H}^n}} f(x),$$

and its dual operator

$$T^*F(x) = \int_{\mathbb{R}} e^{-is\Delta_{\mathbb{H}^n}} F(s, x) \, ds.$$

Then, the estimate (3.18) holds if and only if T is bounded from $L^2(\mathbb{H}^n)$ to $L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)$, or equivalently, if T^* is bounded from $L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{H}^n)$ to $L^2(\mathbb{H}^n)$, or equivalently, if TT^* is bounded from $L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{H}^n)$ to $L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)$, where

$$TT^*F(t, x) = \int_{\mathbb{R}} e^{i(t-s)\Delta_{\mathbb{H}^n}} F(s, x) \, ds.$$

Using the dispersive estimates proved in Theorem 3.4, we have

$$\|e^{i(t-s)\Delta_{\mathbb{H}^n}} F(s, \cdot)\|_{L_x^q} \lesssim_q \left(|t-s|^{-n(\frac{1}{2}-\frac{1}{q})} \mathbf{1}_{\{0 < |t-s| \leq 1\}} + |t-s|^{-\frac{3}{2}} \mathbf{1}_{\{|t-s| > 1\}} \right) \|F(s, \cdot)\|_{L_x^{q'}}.$$

Together with Minkowski's integral inequality, we obtain

$$\|TT^*F(t, \cdot)\|_{L_x^q} \lesssim_q \int_{0 < |t-s| \leq 1} |t-s|^{-n(\frac{1}{2}-\frac{1}{q})} \|F(s, \cdot)\|_{L_x^{q'}} \, ds + \int_{|t-s| > 1} |t-s|^{-\frac{3}{2}} \|F(s, \cdot)\|_{L_x^{q'}} \, ds. \quad (3.19)$$

Set $\alpha = n(1/2 - 1/q)$. Then the condition (3.17) is equivalent to $0 < \alpha \leq 2/p$. Note that, for $0 < \alpha < 2/p$, we have

$$|\tau|^{-\alpha} \mathbf{1}_{\{0 < |\tau| \leq 1\}} \in L^{\frac{p}{2}}(\mathbb{R}).$$

Young's convolution inequality then gives

$$\left\| \int_{|t-s| \leq 1} |t-s|^{-\alpha} \|F(s, \cdot)\|_{L_x^{q'}} \, ds \right\|_{L_t^p} \lesssim \|F\|_{L_t^{p'} L_x^{q'}}. \quad (3.20)$$

If instead $\alpha = 2/p < 1$, the one-dimensional Hardy–Littlewood–Sobolev inequality gives the same bound. ⁵

For the long-time part, since $p \geq 2$ and

$$|\tau|^{-\frac{3}{2}} \mathbf{1}_{\{|\tau| > 1\}} \in L^{\frac{p}{2}}(\mathbb{R}).$$

Using Young's convolution inequality once again, we have

$$\left\| \int_{|t-s| > 1} |t-s|^{-\frac{3}{2}} \|F(s, \cdot)\|_{L_x^{q'}} \, ds \right\|_{L_t^p} \lesssim \|F\|_{L_t^{p'} L_x^{q'}}. \quad (3.21)$$

Combining (3.19), (3.20), and (3.21), we obtain

$$\|TT^*F\|_{L_t^p L_x^q} \lesssim_{p,q} \|F\|_{L_t^{p'} L_x^{q'}}.$$

The proof is therefore complete. $\square$

⁵The excluded endpoint is precisely the remaining case $\alpha = 2/p = 1$, which is not covered by this argument.

We first stated the Strichartz estimate for the free Schrödinger equation in order to keep the main argument transparent. For applications to nonlinear problems, however, the form that is actually needed is the following estimate for the linear Schrödinger equation with a forcing term $F(t, x)$. It can be proved by following the same TT^* and time-convolution argument as in the free case.

Corollary 3.7. *Let (p, q) satisfy the assumptions of Theorem 3.6. Then, the solution $u(t, x)$ of the inhomogeneous Schrödinger equation*

$$\begin{cases} i\partial_t u(t, x) + \Delta_{\mathbb{H}^n} u(t, x) = F(t, x), \\ u(0, x) = u_0(x), \end{cases}$$

satisfies the estimate

$$\|u\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)} \lesssim_{p,q} \|u_0\|_{L^2(\mathbb{H}^n)} + \|F\|_{L_t^{p'} L_x^{q'}(\mathbb{R} \times \mathbb{H}^n)}.$$

3.4 Nonlinear Schrödinger equations

The Strichartz estimates can be used to study the semilinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u = F(u), & (t, x) \in \mathbb{R} \times \mathbb{H}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{H}^n, \end{cases} \quad (3.22)$$

where $F : \mathbb{C} \rightarrow \mathbb{C}$ is a *power-like* nonlinearity of order $\gamma > 1$, that is, there exists a constant $C > 0$ such that, for all $u, v \in \mathbb{C}$,

$$|F(u)| \leq C|u|^\gamma \quad \text{and} \quad |F(u) - F(v)| \leq C(|u|^{\gamma-1} + |v|^{\gamma-1})|u - v|. \quad (3.23)$$

The model nonlinearities $F(u) = \pm|u|^\gamma$ and $F(u) = \pm u|u|^{\gamma-1}$ satisfy these conditions. Notice that neither gauge invariance nor a sign condition is included in the definition.

We recall the Sobolev spaces that will be used below. Let $1 < q < \infty$ and $\sigma \in \mathbb{R}$. The inhomogeneous Sobolev space on $\mathbb{H}^n$ is defined by

$$H^{\sigma,q}(\mathbb{H}^n) := (-\Delta_{\mathbb{H}^n})^{-\frac{\sigma}{2}} L^q(\mathbb{H}^n), \quad \text{with} \quad \|f\|_{H^{\sigma,q}} := \|(-\Delta_{\mathbb{H}^n})^{\frac{\sigma}{2}} f\|_{L^q}.$$

We will write $H^\sigma(\mathbb{H}^n) = H^{\sigma,2}(\mathbb{H}^n)$. We shall use the following Sobolev embedding on hyperbolic spaces.

Lemma 3.8. *Let $1 < q_1 < q_2 < \infty$ and $\sigma_1, \sigma_2 \in \mathbb{R}$. If*

$$\sigma_1 - \sigma_2 \geq n \left(\frac{1}{q_1} - \frac{1}{q_2} \right) > 0,$$

then

$$H^{\sigma_1, q_1}(\mathbb{H}^n) \hookrightarrow H^{\sigma_2, q_2}(\mathbb{H}^n).$$

On hyperbolic spaces, the nonlinear Schrödinger equation (NLS) (3.22) has the following local well-posedness properties. Here *well-posedness* means that a solution exists, is unique in the corresponding solution space, and depends continuously on the initial data.

Theorem 3.9. *If $1 < \gamma < 1 + 4/n$ (resp. $1 < \gamma < 1 + 4/(n-2)$), then the NLS (3.22) is locally well-posed for arbitrary $L^2(\mathbb{H}^n)$ data (resp. $H^1(\mathbb{H}^n)$ data).*

These local results are analogous to their Euclidean counterparts. Indeed, local well-posedness is governed by the behavior of the equation on short time intervals and small spatial scales. In this case, the metric on $\mathbb{H}^n$ is locally Euclidean, and the short-time dispersive estimate has the same decay as on $\mathbb{R}^n$. The advantage of the hyperbolic geometry appears instead in the global-in-time estimates. The enlarged family of admissible Strichartz pairs on $\mathbb{H}^n$ allows the contraction argument to be carried out on the whole time axis for small initial data, leading to the following stronger result.

Theorem 3.10. *If $1 < \gamma \leq 1 + 4/n$ (resp. $1 < \gamma \leq 1 + 4/(n-2)$), then the NLS (3.22) is globally well-posed for small $L^2(\mathbb{H}^n)$ data (resp. $H^1(\mathbb{H}^n)$ data).*

Remark 3.11. In the Euclidean setting, for a general power-like nonlinearity without gauge invariance or a sign condition, small-data global well-posedness is in general available only at the critical exponent. However, on hyperbolic spaces, if F is gauge invariant, then conservation of mass implies that the NLS (3.22) is globally well-posed for arbitrary $L^2(\mathbb{H}^n)$ data in the strictly subcritical range $1 < \gamma < 1 + 4/n$. Similarly, if F is defocusing, then conservation of energy implies that the NLS (3.22) is globally well-posed for arbitrary $H^1(\mathbb{H}^n)$ data for $1 < \gamma < 1 + 4/(n-2)$.

Proof. Recall that a solution of (3.22) satisfies Duhamel's formula

$$u(t, x) = e^{it\Delta_{\mathbb{H}^n}} u_0(x) - i \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}} F(u(s, x)) \, ds.$$

We solve this integral equation by a fixed-point argument on the whole time axis.

We first prove the L^2 statement. Let $p = q = \gamma + 1$. Since $\gamma > 1$, we have $p, q > 2$, and

$$\frac{2}{p} + \frac{n}{q} = \frac{n+2}{\gamma+1} \geq \frac{n}{2}$$

precisely when $\gamma \leq 1 + 4/n$. Thus (p, q) is admissible. Moreover, we have $\gamma p' = p$ and $\gamma q' = q$.

Set

$$X_0 = C(\mathbb{R}; L^2(\mathbb{H}^n)) \cap L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)$$

with norm

$$\begin{aligned} \|v\|_{X_0} &:= \|v\|_{L_t^\infty L_x^2} + \|v\|_{L_t^p L_x^q} \\ &= \sup_{t \in \mathbb{R}} \|v(t, \cdot)\|_{L^2(\mathbb{H}^n)} + \left(\int_{\mathbb{R}} \|v(t, \cdot)\|_{L^q(\mathbb{H}^n)}^p \, dt \right)^{\frac{1}{p}}, \end{aligned}$$

and define

$$\Phi(v)(t) = e^{it\Delta_{\mathbb{H}^n}} u_0 - i \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}} F(v(s)) \, ds.$$

By Strichartz estimate, the power-like assumptions (3.23), and Hölder's inequality,

$$\|\Phi(v)\|_{X_0} \lesssim \|u_0\|_{L^2} + \|F(v)\|_{L_t^{p'} L_x^{q'}} \lesssim \|u_0\|_{L^2} + \|v\|_{X_0}^\gamma, \quad (3.24)$$

while

$$\|\Phi(v) - \Phi(w)\|_{X_0} \lesssim (\|v\|_{X_0}^{\gamma-1} + \|w\|_{X_0}^{\gamma-1}) \|v - w\|_{X_0}. \quad (3.25)$$

Choose $R > 0$ so small that the last coefficient is less than $1/2$ on the ball $\{v \in X_0 : \|v\|_{X_0} \leq R\}$, and then choose $\varepsilon > 0$ so that (3.24) maps this ball into itself whenever $\|u_0\|_{L^2} \leq \varepsilon$. The Banach fixed-point theorem yields a unique global solution

$$u \in C(\mathbb{R}; L^2(\mathbb{H}^n)) \cap L_t^{\gamma+1} L_x^{\gamma+1}.$$

Estimate (3.25) also gives continuous dependence on u_0 . This proves global well-posedness for small L^2 data.

We next turn to the H^1 case. Assume first that $n \geq 3$, and set

$$p = \gamma + 1, \quad q = \frac{n(\gamma + 1)}{n + \gamma - 1}, \quad \text{and} \quad r := \frac{nq}{n - q}.$$

Then

$$\frac{2}{p} + \frac{n}{q} = \frac{n + \gamma + 1}{\gamma + 1} \geq \frac{n}{2} \iff \gamma \leq 1 + \frac{4}{n - 2}.$$

Hence (p, q) is admissible under present assumptions. Moreover, we have the Sobolev embedding

$$H^{1,q}(\mathbb{H}^n) \hookrightarrow L^r(\mathbb{H}^n). \quad (3.26)$$

We know from assumption (3.23) that F is locally Lipschitz:

$$|\nabla_{\mathbb{H}^n} F(v)| \lesssim |v|^{\gamma-1} |\nabla_{\mathbb{H}^n} v| \quad \text{a.e..}$$

Together with Hölder's inequality and (3.26), we obtain

$$\|F(v)\|_{L_t^{p'} H_x^{1,q'}} \lesssim \|v\|_{L_t^p L_x^r}^{\gamma-1} \|v\|_{L_t^p H_x^{1,q}} \lesssim \|v\|_{L_t^p H_x^{1,q}}^\gamma. \quad (3.27)$$

Let $X_1 = C(\mathbb{R}; H^1(\mathbb{H}^n)) \cap L_t^p H_x^{1,q}(\mathbb{R} \times \mathbb{H}^n)$, equipped with its standard norm. Note that the operator $(-\Delta_{\mathbb{H}^n})^{1/2}$ commutes with the propagator $e^{it\Delta_{\mathbb{H}^n}}$. We deduce from Corollary 3.7 that

$$\left\| e^{it\Delta_{\mathbb{H}^n}} v + \int_0^t e^{i(t-s)\Delta_{\mathbb{H}^n}} v(s) \, ds \right\|_{L_t^\infty H_x^1 \cap L_t^p H_x^{1,q}} \lesssim \|v\|_{H^1} + \|v\|_{L_t^{p'} H_x^{1,q'}}.$$

Together with (3.27), we obtain

$$\|\Phi(v)\|_{X_1} \lesssim \|u_0\|_{H^1} + \|v\|_{X_1}^\gamma.$$

To obtain a contraction without imposing any additional differentiability on F , we equip a small ball in X_1 with the weaker metric

$$d(v, w) = \|v - w\|_{L_t^\infty L_x^2} + \|v - w\|_{L_t^p L_x^q}.$$

The assumption (3.23), together with the same Hölder and Sobolev estimates used above, gives

$$d(\Phi(v), \Phi(w)) \lesssim (\|v\|_{X_1}^{\gamma-1} + \|w\|_{X_1}^{\gamma-1}) d(v, w). \quad (3.28)$$

For sufficiently small R and $\|u_0\|_{H^1}$, the map Φ preserves the X_1 -ball of radius R , and its Picard iterates are Cauchy in the metric d . They converge strongly in $L_t^\infty L_x^2 \cap L_t^p L_x^q$ and remain uniformly bounded in $L_t^\infty H_x^1 \cap L_t^p H_x^{1,q}$. Weak-star compactness in $L_t^\infty H_x^1$, together with weak compactness in $L_t^p H_x^{1,q}$, identifies the strong limit with an element of the same X_1 -ball. Thus the limit is a global H^1 solution. Estimate (3.28) gives uniqueness and continuous dependence on the initial data.

When $n = 2$, the upper bound $1 + 4/(n - 2)$ is understood as $+\infty$. For $\gamma > 1$, set

$$p = \gamma + 1, \quad 2 < q < \min \left\{ \gamma + 1, \frac{2(\gamma + 1)}{\gamma - 1} \right\}, \quad \text{and} \quad r := \frac{q(\gamma - 1)}{q - 2}.$$

Then (p, q) is admissible, and $2 < q \leq r < \infty$. The preceding fixed-point argument therefore applies, completing the proof. $\square$

A global L^2 solution u of (3.22) is said to *scatter* if there exist asymptotic states $u_+, u_- \in L^2(\mathbb{H}^n)$ such that

$$\|u(t, \cdot) - e^{it\Delta_{\mathbb{H}^n}} u_\pm\|_{L^2(\mathbb{H}^n)} \longrightarrow 0 \quad \text{as } t \longrightarrow \pm\infty.$$

Scattering means, after possibly nonlinear evolution, the solution approaches free Schrödinger solutions $e^{it\Delta_{\mathbb{H}^n}} u_\pm$ in L^2 as $t \rightarrow \pm\infty$. Intuitively, as the wave disperses over an increasingly large spatial region, its amplitude in any fixed region becomes small, and hence the nonlinear interaction becomes negligible at large times. Scattering in H^1 is defined by replacing L^2 with H^1 in the preceding limits.

Theorem 3.12. *If $1 < \gamma \leq 1 + 4/n$ (resp. $1 < \gamma \leq 1 + 4/(n - 2)$), then the global solution of NLS (3.22), corresponding to small initial data, scatters in $L^2(\mathbb{H}^n)$ (resp. $H^1(\mathbb{H}^n)$).*

Proof. Keep the exponents and notation from the proof of Theorem 3.10, and set

$$z(t) := e^{-it\Delta_{\mathbb{H}^n}} u(t).$$

Since the Schrödinger propagator is unitary, scattering is equivalent to the existence of the limits of $z(t)$ as $t \rightarrow \pm\infty$ in the corresponding Sobolev space.

In the L^2 case, take $p = q = \gamma + 1$, we know from Theorem 3.10 that there exists a unique global solution $u \in C(\mathbb{R}; L^2(\mathbb{H}^n)) \cap L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)$. For $t_1 < t_2$, Duhamel's formula yields

$$z(t_2) - z(t_1) = -i \int_{t_1}^{t_2} e^{-is\Delta_{\mathbb{H}^n}} F(u(s)) ds.$$

The dual homogeneous Strichartz estimate, followed by the estimate (3.24), gives

$$\|z(t_2) - z(t_1)\|_{L^2} \lesssim \|F(u)\|_{L_t^{p'}([t_1, t_2]; L_x^{q'})} \lesssim \|u\|_{L_t^p([t_1, t_2]; L_x^q)}^\gamma.$$

Since $u \in L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^n)$, the last quantity tends to zero when $t_1, t_2 \rightarrow \pm\infty$. Thus $z(t)$ is Cauchy in $L^2(\mathbb{H}^n)$ at both ends of the time axis. Since $L^2(\mathbb{H}^n)$ is complete, there exist $u_\pm \in L^2(\mathbb{H}^n)$ such that

$$z(t) \longrightarrow u_\pm \quad \text{in } L^2(\mathbb{H}^n) \quad \text{as } t \longrightarrow \pm\infty.$$

By unitarity once more,

$$\|u(t) - e^{it\Delta_{\mathbb{H}^n}} u_\pm\|_{L^2} = \|z(t) - u_\pm\|_{L^2} \longrightarrow 0, \quad \text{as } t \longrightarrow \pm\infty,$$

which proves scattering in L^2 . The proof of scattering in H^1 is similar, and is therefore omitted. $\square$

Remark 3.13. Recall that, in the Euclidean setting, the H^1 scattering for the NLS with model nonlinearity $F(u) = |u|^{\gamma-1}u$ is known to fail for $1 < \gamma < 1 + 2/n$.

IV On Symmetric Spaces of Non-Compact Type and Higher Rank

The rank of a symmetric space is formulated in terms of its flats. A *flat* in a simply connected Riemannian manifold X is a complete, connected, totally geodesic submanifold $\mathcal{F} \subset X$ with zero sectional curvature.⁶ Equivalently, if $\ell = \dim \mathcal{F}$, then

$$\mathcal{F} \simeq \mathbb{R}^\ell,$$

with respect to its induced Riemannian metric. Here *totally geodesic* means that every geodesic of $\mathcal{F}$ is also a geodesic of X . A flat is called *maximal* if it is not properly contained in any larger flat. In a symmetric space of non-compact type, all maximal flats have the same dimension. This common dimension is the rank of X :

$$\ell := \text{rank } X = \dim \mathcal{F}, \quad \mathcal{F} \text{ a maximal flat.}$$

Example 4.1. Every complete geodesic in real hyperbolic space $\mathbb{H}^n$ is a one-dimensional flat. Since every tangent plane in $\mathbb{H}^n$ has sectional curvature -1 , no two-dimensional flat can occur. Therefore, $\mathbb{H}^n$ is a symmetric space of rank one. In contrast, consider two complete geodesics $\gamma_1 \subset \mathbb{H}^{n_1}$ and $\gamma_2 \subset \mathbb{H}^{n_2}$. Then $\gamma_1 \times \gamma_2 \simeq \mathbb{R}^2$ is a two-dimensional flat in $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$, representing two independent geodesic directions. Thus $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ is a symmetric space of rank two.

A principal geometric distinction between rank one and higher rank is therefore the presence of higher-dimensional Euclidean flats. In rank one, every maximal flat is a geodesic, and the position of a point, modulo the isometries fixing a base point, is determined by a single distance. In higher rank, several independent flat directions are present, and distance alone no longer determines the position of a point modulo these isometries. The resulting geometry is *anisotropic*.

4.1 Root systems

Let $X = G/K$ be a Riemannian symmetric space of non-compact type, where G is a connected non-compact semisimple Lie group with finite center, and K is its maximal compact subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K . Write the Cartan decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$, identified with the tangent space $T_o X$ at the base point $o \in X$. Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Then $\mathfrak{a}$ is the tangent space at o of a maximal flat $\mathcal{F}$:

$$\mathcal{F} = \exp_o(\mathfrak{a}) \simeq \mathbb{R}^{\dim \mathfrak{a}}.$$

The rank of X is therefore $\ell = \dim \mathfrak{a}$.

The spectral parameter of the spherical Fourier transform belongs to the dual space $\mathfrak{a}^*$, so the frequency space has the same dimension as the rank. For real hyperbolic space, $\ell = 1$, and the spectral parameter is a single real number. In higher rank, $\ell \geq 2$, and the spectral parameter is a vector in the Euclidean space $\mathfrak{a}^* \simeq \mathbb{R}^\ell$. Although $\mathfrak{a}$ and $\mathfrak{a}^*$ are Euclidean spaces, they carry distinguished directions and hyperplanes determined by the geometry of X . These data are organized by a *root system*.

⁶In dimension one, every complete geodesic line is a one-dimensional flat.

Definition 4.2. Let V be a finite-dimensional Euclidean vector space. A finite set $\Sigma \subset V \setminus \{0\}$ is a crystallographic root system if

- (i) Σ spans V ;
- (ii) for every $\alpha \in \Sigma$, the orthogonal reflection

$$s_\alpha(v) = v - 2 \frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

preserves the set of roots: $s_\alpha(\Sigma) = \Sigma$;

- (iii) for all $\alpha, \beta \in \Sigma$,

$$2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

A root system is called reduced if the only roots on the line $\mathbb{R}\alpha$ are $\pm\alpha$.

For $\alpha \in \mathfrak{a}^*$, we define the root space by

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : [H, Y] = \alpha(H)Y \text{ for every } H \in \mathfrak{a}\}.$$

The finite set

$$\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$$

is called the *restricted root system* of X , and $m_\alpha = \dim \mathfrak{g}_\alpha$ is the multiplicity of α . After choosing positive roots Σ^+ , the corresponding simple roots $\alpha_1, \dots, \alpha_\ell$ determine the positive Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha_j(H) > 0, 1 \leq j \leq \ell\}.$$

Its boundary hyperplanes $\ker \alpha_j = \{H \in \mathfrak{a} : \alpha_j(H) = 0\}$ are called the *walls*. The group generated by reflections across these walls is the Weyl group W . In rank one, the positive Weyl chamber is only a half-line and its sole wall is the origin. In higher rank it is a cone with several walls, so two frequency vectors of the same length may have different positions relative to the walls. This is the basic distinction that will matter for estimates of the spherical functions and $\mathbf{c}$ -function.

Example 4.3. For real hyperbolic space $\mathbb{H}^n$, the restricted root system is reduced:

$$\Sigma = \{\pm\alpha\}, \quad m_\alpha = n - 1.$$

It is the one-dimensional root system of type A_1 . For the other rank-one symmetric spaces of non-compact type, namely complex hyperbolic space, quaternionic hyperbolic space, and the Cayley hyperbolic plane, the restricted root system is non-reduced:

$$\Sigma = \{\pm\alpha, \pm 2\alpha\},$$

except for some low-dimensional exceptions, which are isomorphic to A_1 .

Example 4.4. Up to isomorphism, the reduced root systems of rank two are

$$D_2 \simeq A_1 \times A_1, \quad A_2, \quad B_2 \simeq C_2, \quad G_2.$$

They contain respectively 4, 6, 8, and 12 roots, and their positive Weyl chambers have angles

$$\frac{\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{\pi}{4}, \quad \frac{\pi}{6}.$$

All roots have the same length in types D_2 and A_2 . Types B_2 and G_2 have two root lengths, with long-to-short length ratios $\sqrt{2}$ and $\sqrt{3}$, respectively.

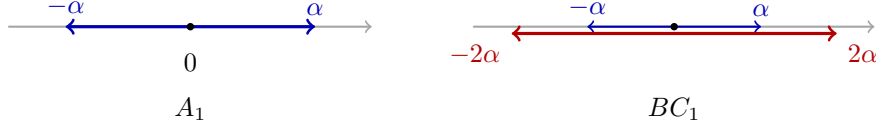


Figure 3: The reduced rank-one root system A_1 and the non-reduced rank-one root system BC_1 .

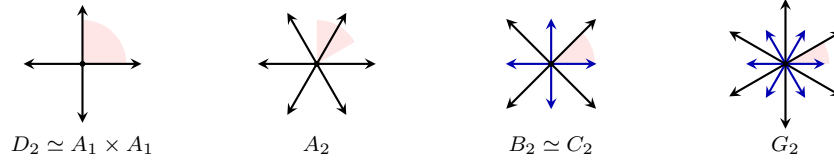


Figure 4: Root systems of rank two. The positive Weyl chamber is shaded in red.

Example 4.5. Consider $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, which can be identified with the space of positive-definite symmetric $n \times n$ matrices of determinant one. X is a symmetric space of non-compact type with rank $\ell = n - 1$ and dimension $(n - 1)(n + 2)/2$. For $n = 3$, the Cartan subspace is

$$\mathfrak{a} = \{\mathrm{diag}(r_1, r_2, r_3) : r_1 + r_2 + r_3 = 0\}.$$

For $i \neq j$, define

$$\alpha_{ij}(H) = r_i - r_j, \quad H = \mathrm{diag}(r_1, r_2, r_3).$$

Then

$$\Sigma = \{\alpha_{ij} : 1 \leq i \neq j \leq 3\}$$

is the root system of type A_2 , and every root has multiplicity one. We may choose $\alpha = \alpha_{12}$ and $\beta = \alpha_{23}$ as the simple roots. The set of positive roots is

$$\Sigma^+ = \{\alpha, \beta, \alpha + \beta\},$$

and the positive Weyl chamber is

$$\mathfrak{a}^+ = \{\mathrm{diag}(r_1, r_2, r_3) \in \mathfrak{a} : r_1 > r_2 > r_3\}.$$

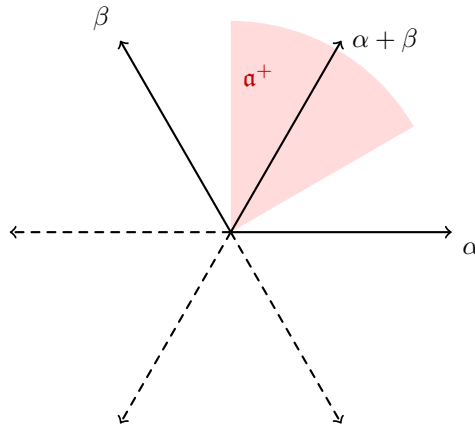


Figure 5: The root system A_2 and its positive Weyl chamber.

4.2 The $\mathbf{c}$ -function in higher rank

The spherical Fourier transform is W -invariant in the spectral variable. Its frequency space is therefore the quotient $\mathfrak{a}^*/W$, which may be represented by the closed positive Weyl chamber $\overline{\mathfrak{a}}_+^*$. For real hyperbolic space this frequency space is one-dimensional. In higher rank the spectral parameter has several components and may approach infinity while remaining close to one or more Weyl walls.

Let

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \quad \text{and} \quad \bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}.$$

Denote by $N = \exp \mathfrak{n}$ and by $\bar{N} = \exp \bar{\mathfrak{n}}$. According to the Iwasawa decomposition $G = KAN$, every $g \in G$ may be written uniquely as

$$g = k(g) \exp H(g) n(g), \quad H(g) \in \mathfrak{a}.$$

Up to the normalization of Haar measure on $\bar{N}$, the Harish–Chandra $\mathbf{c}$ -function is defined by

$$\mathbf{c}(\lambda) = \int_{\bar{N}} e^{-(i\lambda + \rho)(H(\bar{n}))} d\bar{n},$$

where $\rho \in \mathfrak{a}^*$ is the half-sum of positive roots counted with multiplicities:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

The following Gindikin–Karpelevich formula reduces the preceding multidimensional integral to a product of one-dimensional Gamma factors.

Proposition 4.6. *Let $\Sigma_r^+ = \{\alpha \in \Sigma^+ : \alpha/2 \notin \Sigma\}$ be the set of positive reduced roots. For $\lambda \in \mathfrak{a}^*$,*

$$\mathbf{c}(\lambda) = \prod_{\alpha \in \Sigma_r^+} \mathbf{c}_\alpha \left(\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \right),$$

where

$$\mathbf{c}_\alpha(v) = \frac{\Gamma\left(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{2}m_\alpha\right)}{\Gamma\left(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle}\right)} \frac{\Gamma\left(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}\right)}{\Gamma\left(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha\right)} \frac{\Gamma(iv)}{\Gamma(iv + \frac{1}{2}m_\alpha)} \frac{\Gamma(\frac{i}{2}v + \frac{1}{4}m_\alpha)}{\Gamma(\frac{i}{2}v + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}.$$

We deduce from the Gindikin–Karpelevich formula that the Plancherel density of the spherical Fourier transform satisfies

$$|\mathbf{c}(\lambda)|^{-2} \asymp \prod_{\alpha \in \Sigma_r^+} |\langle \alpha, \lambda \rangle|^2 (1 + |\langle \alpha, \lambda \rangle|)^{m_\alpha + m_{2\alpha} - 2}.$$

In rank one there is only one root variable, so the size of $|\langle \alpha, \lambda \rangle|$ is comparable to $|\lambda|$. In higher rank, however, the density also depends on the direction of λ , not only on its length. $|\lambda|$ may be large while $|\langle \alpha, \lambda \rangle|$ remains small for a particular root α . Geometrically, λ is then close to the corresponding Weyl wall.

As a consequence, near a wall, differentiation in the direction normal to that wall need not produce the expected factor $(1 + |\lambda|)^{-1}$. Therefore the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ is generally not a differential symbol on $\mathfrak{a}^*$. Without global symbol bounds, the standard stationary-phase method cannot be applied uniformly across the whole Weyl chamber. The *barycentric decomposition* introduced next allows one to overcome this difficulty. It divides the chamber into subcones on which one can choose a direction of differentiation adapted to the relevant root factors. Then, in each subcone, $|\mathbf{c}(\lambda)|^{-2}$ behaves as if it were a symbol along this well-chosen direction.

4.3 Barycentric decomposition of the Weyl chamber

Throughout this subsection we consider the higher-rank case $\ell \geq 2$. Let $\Sigma_s^+ = \{\alpha_1, \dots, \alpha_\ell\}$ be the set of positive simple roots. Let $\{\Lambda_1, \dots, \Lambda_\ell\} \subset \mathfrak{a}$ be the dual basis, defined by

$$\langle \alpha_j, \Lambda_k \rangle = \delta_{jk}, \quad 1 \leq j, k \leq \ell.$$

Note that

$$\begin{cases} \langle \alpha_j, \alpha_k \rangle \leq 0, & j \neq k, \\ \langle \Lambda_j, \Lambda_k \rangle \geq 0, & 1 \leq j, k \leq \ell. \end{cases} \quad (4.1)$$

Since the simple roots $\alpha_1, \dots, \alpha_\ell$ form a basis of $\mathfrak{a}^*$, the dual vectors $\Lambda_1, \dots, \Lambda_\ell$ form a basis of $\mathfrak{a}$. Writing

$$H = \sum_{k=1}^{\ell} c_k \Lambda_k$$

and pairing both sides with α_j , we obtain

$$\langle \alpha_j, H \rangle = \sum_{k=1}^{\ell} c_k \langle \alpha_j, \Lambda_k \rangle = \sum_{k=1}^{\ell} c_k \delta_{jk} = c_j.$$

Thus every $H \in \mathfrak{a}$ has the unique expansion

$$H = \sum_{k=1}^{\ell} \langle \alpha_k, H \rangle \Lambda_k, \quad (4.2)$$

and $\overline{\mathfrak{a}^+} = \mathbb{R}_+ \Lambda_1 + \dots + \mathbb{R}_+ \Lambda_\ell$.

Let $\mathfrak{B}$ be the convex hull of $W\Lambda_1 \cup \dots \cup W\Lambda_\ell$, and let $\mathfrak{S} = \partial\mathfrak{B}$ be its polyhedral boundary. Then $\mathfrak{B} \cap \overline{\mathfrak{a}^+}$ is the ℓ -simplex with vertices $0, \Lambda_1, \dots, \Lambda_\ell$, whereas $\mathfrak{S} \cap \overline{\mathfrak{a}^+}$ is the $(\ell - 1)$ -simplex with vertices $\Lambda_1, \dots, \Lambda_\ell$. For $1 \leq j \leq \ell$, define

$$\mathfrak{S}_j = \left\{ \lambda \in \mathfrak{S} \cap \overline{\mathfrak{a}^+} : \langle \alpha_j, \lambda \rangle = \max_{1 \leq k \leq \ell} \langle \alpha_k, \lambda \rangle \right\}.$$

The barycentric subdivisions of the simplices $\mathfrak{S} \cap w\overline{\mathfrak{a}^+}$ give the tiling

$$\mathfrak{S} = \bigcup_{w \in W} \bigcup_{j=1}^{\ell} w\mathfrak{S}_j. \quad (4.3)$$

The tile $\mathfrak{S}_j$ is therefore the part of the positive simplex on which the j -th barycentric coordinate is the largest.

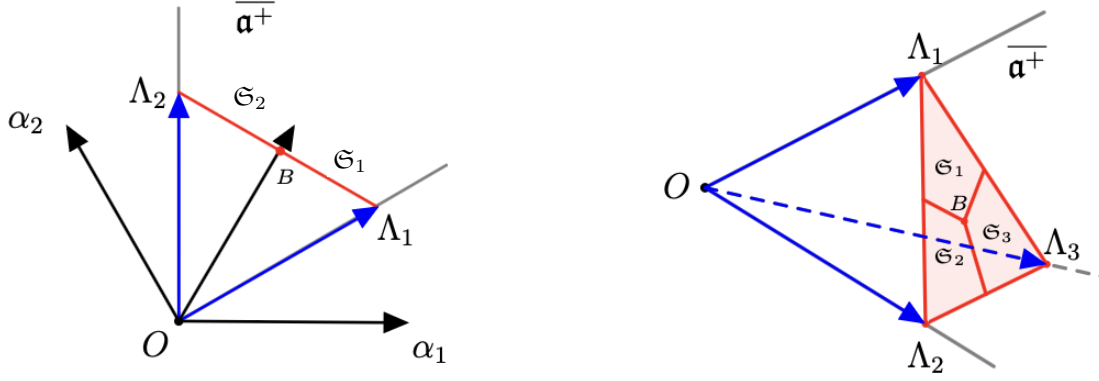


Figure 6: Examples of barycentric subdivisions in types A_2 and A_3 .

Remark 4.7. The tile $\mathfrak{S}_j$ is the convex hull of all points

$$\frac{\Lambda_{k_1} + \cdots + \Lambda_{k_r}}{r},$$

where $\{\Lambda_{k_1}, \dots, \Lambda_{k_r}\}$ runs through the nonempty subsets of $\{\Lambda_1, \dots, \Lambda_\ell\}$ which contain Λ_j . These points are the barycenters of the faces containing the vertex Λ_j .

Lemma 4.8. *Let $w \in W$ and $1 \leq j \leq \ell$. Then:*

(i) *A root $\alpha \in \Sigma$ is orthogonal to some vector in the tile $w\mathfrak{S}_j$ if and only if it is orthogonal to the vertex $w\Lambda_j$.*

(ii) *For every $\lambda \in w\mathfrak{S}_j$,*

$$\langle w\Lambda_j, \lambda \rangle \geq \frac{1}{\ell} |\Lambda_j|^2.$$

Proof. For (i), the implication from the vertex to the tile is immediate because $w\Lambda_j \in w\mathfrak{S}_j$. We prove the converse. It is enough to consider $w = \text{id}$ and $\alpha \in \Sigma^+$.

Since every positive root is a nonnegative integral combination of the simple roots,

$$\alpha = \sum_{k=1}^{\ell} \langle \alpha, \Lambda_k \rangle \alpha_k, \quad \langle \alpha, \Lambda_k \rangle \in \mathbb{N}. \quad (4.4)$$

For $\lambda \in \mathfrak{S} \cap \overline{\mathfrak{a}^+}$, formula (4.2) reads

$$\lambda = \sum_{k=1}^{\ell} \langle \alpha_k, \lambda \rangle \Lambda_k.$$

Here the coefficients are nonnegative and add up to 1, because λ lies in the simplex with vertices $\Lambda_1, \dots, \Lambda_\ell$. They are therefore precisely the barycentric coordinates of λ .

If $\lambda \in \mathfrak{S}_j$, its j -th coordinate is the largest one. In particular,

$$\langle \alpha_j, \lambda \rangle \geq \frac{1}{\ell} > 0.$$

Using (4.4), we obtain

$$\langle \alpha, \lambda \rangle = \sum_{k=1}^{\ell} \underbrace{\langle \alpha, \Lambda_k \rangle}_{\geq 0} \underbrace{\langle \alpha_k, \lambda \rangle}_{\geq 0}.$$

If this sum vanishes, then its j -th term vanishes. Since $\langle \alpha_j, \lambda \rangle > 0$, it follows that $\langle \alpha, \Lambda_j \rangle = 0$, proving (i).

For (ii), we again take $w = \text{id}$. For every $\lambda \in \mathfrak{S} \cap \overline{\mathfrak{a}^+}$, taking the inner product of (4.2) with Λ_j , and using (4.1), we obtain

$$\begin{aligned} \langle \Lambda_j, \lambda \rangle &= \sum_{k=1}^{\ell} \langle \Lambda_j, \Lambda_k \rangle \langle \alpha_k, \lambda \rangle \\ &= |\Lambda_j|^2 \underbrace{\langle \alpha_j, \lambda \rangle}_{\geq \frac{1}{\ell}} + \sum_{k \neq j} \underbrace{\langle \Lambda_j, \Lambda_k \rangle}_{\geq 0} \underbrace{\langle \alpha_k, \lambda \rangle}_{\geq 0} \\ &\geq \frac{1}{\ell} |\Lambda_j|^2. \end{aligned}$$

This completes the proof. □

We now project the preceding polyhedral tiling radially onto the unit sphere in $\mathfrak{a}$. Set

$$S(\mathfrak{a}) = \{\lambda \in \mathfrak{a} : |\lambda| = 1\} \quad \text{and} \quad S_j = \left\{ \frac{\lambda}{|\lambda|} : \lambda \in \mathfrak{S}_j \right\}.$$

Since every nonzero vector meets $\mathfrak{S}$ in exactly one point after a positive rescaling, (4.3) yields

$$S(\mathfrak{a}) = \bigcup_{w \in W} \bigcup_{j=1}^{\ell} w S_j.$$

Thus each nonzero frequency belongs, after normalization, to at least one barycentric tile. Figure 7 illustrates this radial projection in type A_3 .

Apart from their common boundaries, which have measure zero, the projected tiles have disjoint interiors. Consequently, their characteristic functions form the partition

$$\sum_{w \in W} \sum_{j=1}^{\ell} \mathbf{1}_{w S_j} \left(\frac{\lambda}{|\lambda|} \right) = 1 \quad \text{for almost every } \lambda \in \mathfrak{a} \setminus \{0\}.$$

These characteristic functions are not smooth. We now construct a smooth version of this partition for later differentiation in oscillatory integrals. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be smooth and satisfy

$$\chi(s) = 1 \quad \text{for } s \geq 0, \quad \chi(s) = 0 \quad \text{for } s \leq -c_1,$$

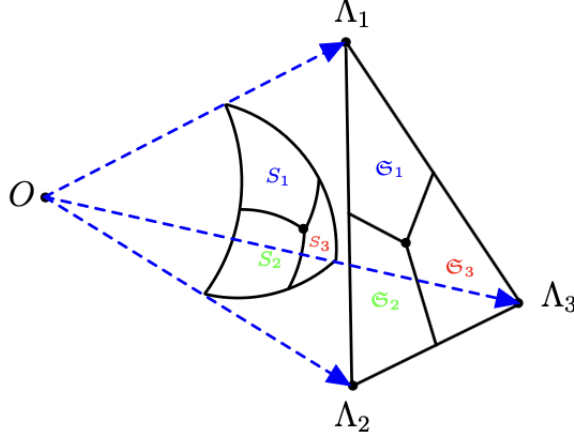


Figure 7: Radial projection of the barycentric subdivision in type A_3 .

where $c_1 > 0$ will be fixed below. For $w \in W$, $1 \leq j \leq \ell$, and $\lambda \neq 0$, define

$$\tilde{\chi}_{wS_j}(\lambda) = \prod_{\substack{1 \leq k \leq \ell \\ k \neq j}} \chi\left(\frac{\langle w\alpha_k, \lambda \rangle}{|\lambda|}\right) \chi\left(\frac{\langle w\alpha_j, \lambda \rangle - \langle w\alpha_k, \lambda \rangle}{|\lambda|}\right), \quad (4.5)$$

and set

$$\tilde{\chi}(\lambda) = \sum_{w \in W} \sum_{j=1}^{\ell} \tilde{\chi}_{wS_j}(\lambda). \quad (4.6)$$

Proposition 4.9. *For every $w \in W$ and $1 \leq j \leq \ell$, the functions above have the following properties on $\mathfrak{a} \setminus \{0\}$:*

- (i) $\tilde{\chi}_{wS_j}(w\lambda) = \tilde{\chi}_{S_j}(\lambda)$, and $\tilde{\chi}$ is W -invariant.
- (ii) $\tilde{\chi}_{wS_j} = 1$ on wS_j , and $\tilde{\chi} \geq 1$.
- (iii) The functions $\tilde{\chi}_{wS_j}$ and $\tilde{\chi}$ are homogeneous symbols of order zero.

Proof. The Weyl group preserves the inner product: $\langle w\alpha_k, w\lambda \rangle = \langle \alpha_k, \lambda \rangle$, which proves (i).

For (ii), by symmetry it suffices to take $w = \text{id}$. If $\lambda \in S_j$, then $\langle \alpha_k, \lambda \rangle \geq 0$, and

$$\langle \alpha_j, \lambda \rangle \geq \langle \alpha_k, \lambda \rangle, \quad k \neq j.$$

Every factor in (4.5) is therefore equal to one. Since the tiles cover the unit sphere, and all the cutoffs are homogeneous of degree zero, at least one summand in (4.6) equals one at every nonzero λ . Thus $\tilde{\chi} \geq 1$. (iii) is directly verified from their constructions. $\square$

We normalize the preceding cutoffs by setting

$$\chi_{wS_j}(\lambda) = \frac{\tilde{\chi}_{wS_j}(\lambda)}{\tilde{\chi}(\lambda)}.$$

Proposition 4.9 shows that these are homogeneous symbols of order zero and satisfy

$$\sum_{w \in W} \sum_{j=1}^{\ell} \chi_{wS_j}(\lambda) = 1, \quad \lambda \in \mathfrak{a} \setminus \{0\}.$$

The supports of these functions are slightly larger than the original tiles. The vectors inside satisfy further properties. To prove them, we need the following lemma.

Lemma 4.10. *There exists $c_2 > 0$ with the following property. If, for some $1 \leq j \leq \ell$, a vector $\lambda \in \mathfrak{a}$ satisfies*

$$-c_2|\lambda| \leq \langle \alpha_k, \lambda \rangle \leq \langle \alpha_j, \lambda \rangle + c_2|\lambda|, \quad k \neq j,$$

then $\langle \alpha_j, \lambda \rangle \geq c_2|\lambda|$.

Proof. By homogeneity, it suffices to consider $|\lambda| = 1$. Since the simple roots form a basis of $\mathfrak{a}^*$, the function

$$\lambda \mapsto \sum_{k=1}^{\ell} |\langle \alpha_k, \lambda \rangle|$$

is a norm on $\mathfrak{a}$. Its minimum on the unit sphere is therefore positive. Thus there exists $c_3 > 0$ such that, for all $\lambda \in S(\mathfrak{a})$ with $|\lambda| = 1$,

$$\sum_{k=1}^{\ell} |\langle \alpha_k, \lambda \rangle| \geq c_3. \quad (4.7)$$

Set $c_2 = c_3/(2\ell)$. Suppose first that $\langle \alpha_k, \lambda \rangle \leq 2c_2$ for every $k \neq j$. If $\langle \alpha_j, \lambda \rangle < 2c_2$, then the hypotheses also give $\langle \alpha_j, \lambda \rangle \geq -2c_2$, while $-c_2 \leq \langle \alpha_k, \lambda \rangle \leq 2c_2$ for $k \neq j$. Consequently,

$$\sum_{k=1}^{\ell} |\langle \alpha_k, \lambda \rangle| < 2\ell c_2 = c_3,$$

contradicting (4.7). Hence in this case $\langle \alpha_j, \lambda \rangle \geq 2c_2$.

Otherwise, for some $k \neq j$, $\langle \alpha_k, \lambda \rangle \geq 2c_2$. The upper inequality in the hypothesis then gives

$$\langle \alpha_j, \lambda \rangle \geq \langle \alpha_k, \lambda \rangle - c_2 \geq c_2.$$

This proves the lemma. □

Remark 4.11. We record the constants needed below. Let

$$L_1 = \max_{\alpha \in \Sigma^+} \sum_{k=1}^{\ell} \langle \alpha, \Lambda_k \rangle \quad \text{and} \quad L_2 = \sum_{k=1}^{\ell} |\Lambda_k|.$$

Thus L_1 is the maximum height of a positive root with respect to the chosen simple roots. Set also

$$M_1 = \min_{1 \leq k \leq \ell} |\Lambda_k| \quad \text{and} \quad M_2 = \max_{1 \leq k \leq \ell} |\Lambda_k|.$$

Choose the constant in the cutoff above so that

$$0 < c_1 < c_2 \min \left\{ \frac{1}{L_1}, \frac{M_1^2}{M_2 L_2} \right\},$$

and define

$$c_4 = c_2 - L_1 c_1 \quad \text{and} \quad c_5 = M_1^2 c_2 - M_2 L_2 c_1.$$

Then $c_4, c_5 > 0$, and all these constants depend only on the root system.

The following result is the quantitative form of Lemma 4.8 for the enlarged supports of the smooth partition.

Proposition 4.12. *Let $w \in W$ and $1 \leq j \leq \ell$. Then:*

(i) *For every root $\alpha \in \Sigma$, either $\langle \alpha, w\Lambda_j \rangle = 0$, or*

$$|\langle \alpha, \lambda \rangle| \geq c_4 |\lambda| \quad \text{for every } \lambda \in \text{supp } \chi_{wS_j}.$$

(ii) *For every $\lambda \in \text{supp } \chi_{wS_j}$,*

$$|\langle w\Lambda_j, \lambda \rangle| \geq c_5 |\lambda|.$$

Proof. Again, by symmetry it suffices to take $w = \text{id}$, and assume $\alpha \in \Sigma^+$. If $\lambda \in \text{supp } \chi_{S_j}$, the factors in (4.5) imply

$$-c_1 |\lambda| \leq \langle \alpha_k, \lambda \rangle \leq \langle \alpha_j, \lambda \rangle + c_1 |\lambda|, \quad k \neq j. \quad (4.8)$$

Indeed, outside these inequalities at least one factor χ vanishes. Since $c_1 < c_2$, Lemma 4.10 gives

$$\langle \alpha_j, \lambda \rangle \geq c_2 |\lambda|. \quad (4.9)$$

To prove (i), suppose that $\langle \alpha, \Lambda_j \rangle > 0$. The expansion (4.4) and integrality of its coefficients give $\langle \alpha, \Lambda_j \rangle \geq 1$. Combining (4.8) and (4.9), we obtain

$$\begin{aligned} \langle \alpha, \lambda \rangle &= \sum_{k=1}^{\ell} \langle \alpha, \Lambda_k \rangle \langle \alpha_k, \lambda \rangle \\ &\geq c_2 |\lambda| - c_1 |\lambda| \sum_{k \neq j} \langle \alpha, \Lambda_k \rangle \\ &\geq (c_2 - L_1 c_1) |\lambda| = c_4 |\lambda|. \end{aligned}$$

This proves (i).

For (ii), taking the inner product of (4.2) with Λ_j then using (4.1), (4.8), and (4.9), gives

$$\begin{aligned} \langle \Lambda_j, \lambda \rangle &= \sum_{k=1}^{\ell} \langle \Lambda_j, \Lambda_k \rangle \langle \alpha_k, \lambda \rangle \\ &\geq |\Lambda_j|^2 c_2 |\lambda| - c_1 |\lambda| \sum_{k \neq j} \langle \Lambda_j, \Lambda_k \rangle \\ &\geq (M_1^2 c_2 - M_2 L_2 c_1) |\lambda| = c_5 |\lambda|, \end{aligned}$$

for all $\lambda \in \text{supp } \chi_{S_j}$. □

Corollary 4.13. *There exists a smooth partition of unity*

$$\sum_{w \in W} \sum_{j=1}^{\ell} \chi_{wS_j}(\lambda) = 1, \quad \lambda \in \mathfrak{a} \setminus \{0\},$$

consisting of nonnegative homogeneous symbols of order zero. For every $w \in W$ and $1 \leq j \leq \ell$, the nonzero vector $w\Lambda_j$ belongs to $\text{supp } \chi_{wS_j}$, and

$$|\langle w\Lambda_j, \lambda \rangle| \asymp |\lambda|, \quad \lambda \in \text{supp } \chi_{wS_j}.$$

Moreover, for every root $\alpha \in \Sigma$, the following dichotomy holds:

- (i) *either $\langle \alpha, w\Lambda_j \rangle = 0$;*
- (ii) *or $|\langle \alpha, \lambda \rangle| \asymp |\lambda|$, for $\lambda \in \text{supp } \chi_{wS_j}$.*

All comparison constants may be chosen uniformly in w , j , and α .

Finally, we record the following consequence of the barycentric decomposition. In this way, we can treat the derivatives of the Plancherel density, and get precise estimates in higher rank.

Corollary 4.14. *For every $w \in W$, $1 \leq j \leq \ell$, and $k \in \mathbb{N}$,*

$$\partial_{w\Lambda_j}^k |\mathbf{c}(\lambda)|^{-2} = O(|\lambda|^{n-\ell-k}), \quad \forall \lambda \in \text{supp } \chi_{wS_j}. \quad (4.10)$$

Proof. Using the Gindikin–Karpelevich formula, we write the Plancherel density as the product

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha \in \Sigma_r^+} \left| \mathbf{c}_\alpha \left(\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \right) \right|^{-2}.$$

For each $\alpha \in \Sigma_r^+$, the corresponding factor is an inhomogeneous one-dimensional differential symbol of order $m_\alpha + m_{2\alpha}$. If $\langle \alpha, w\Lambda_j \rangle = 0$, then every directional derivative along $w\Lambda_j$ of this factor vanishes. Otherwise, Corollary 4.13 gives

$$|\langle \alpha, \lambda \rangle| \asymp |\lambda| \quad \text{on } \text{supp } \chi_{wS_j}.$$

Consequently, for every integer $k \geq 0$, differentiation of this factor k times in the direction $w\Lambda_j$ gives the bound

$$O(|\lambda|^{m_\alpha + m_{2\alpha} - k}) \quad (|\lambda| \geq 1).$$

The estimate (4.10) then follows from the fact that

$$n - \ell = \sum_{\alpha \in \Sigma_r^+} (m_\alpha + m_{2\alpha}).$$

The proof is therefore complete. □

References

Partly out of laziness, the author has not inserted precise references throughout every section. The principal sources used in preparing these lecture notes are collected below.

For Schrödinger equations on Euclidean spaces, we refer to the classical texts:

- T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003, xiv+323 pp.
- T. Tao, *Nonlinear dispersive equations. Local and global analysis*, CBMS Regional Conference Series in Mathematics, 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, xvi+373 pp.

For harmonic analysis on Lie groups and symmetric spaces, we refer to:

- S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York–London, 1978, xv+628 pp.
- S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, corrected reprint of the 1984 original, Mathematical Surveys and Monographs, 83, American Mathematical Society, Providence, RI, 2000, xxii+667 pp., doi: [10.1090/surv/083](https://doi.org/10.1090/surv/083).

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- J.-Ph. Anker and V. Pierfelice, Nonlinear Schrödinger equation on real hyperbolic spaces, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), no. 5, 1853–1869, doi:[10.1016/j.anihp.2009.01.009](https://doi.org/10.1016/j.anihp.2009.01.009).
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- J.-Ph. Anker and H.-W. Zhang, Wave equation on general noncompact symmetric spaces, *Amer. J. Math.* **146** (2024), no. 4, 983–1031, doi:[10.1353/ajm.2024.a932434](https://doi.org/10.1353/ajm.2024.a932434).
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